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$D = 2$, $\mathcal{N} = 2$, supersymmetric theories on non(anti)commutative superspace

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Abstract: The classical action of a two dimensional $\mathcal{N} = 2$ supersymmetric theory, characterized by a general Kähler potential, is written down on a non(anti)commutative superspace. The action has a power series expansion in terms of the determinant of the non(anti)commutativity parameter $C^{\alpha\beta}$. The theory is explicitly shown to preserve half of the $\mathcal{N} = 2$ supersymmetry, to all orders in $(\det C)^n$. The results are further generalized to include arbitrary superpotentials as well.

Keywords: Field Theories in Lower Dimensions, Sigma Models, Extended Supersymmetry, Superspaces.

Contents

1. Introduction

Field theories on noncommutative spaces have been studied extensively in the past few years (for a lucid review and full list of references, see [1]), even more so, after their appearance in certain limits of String and M-theories [2]–[4]. Noncommutativity of the bosonic space-time coordinates emerges on a D-brane worldvolume, when a constant NS-NS two form field is turned on. The study of classical, as well as the quantum aspects of noncommutative field theories has thrown in many surprises, both at the perturbative and nonperturbative level. The most notable ones being, UV/IR mixing [5, 6], noncommutative solitons [7], quantum Hall fluid [8] etc.

Considering the emergence of such interesting concepts, it is natural to seek a generalization of the idea of noncommutativity to be applicable to wider situations. An obvious thing to check is the possibility of having non(anti)commutativity between the superspace coordinates. Indeed, in the context of Dijkgraaf-Vafa correspondence [9] relating $\mathcal{N} = 1$ supersymmetric gauge theories and matrix models, it was suggested [10]–[13], that non(anti)commutativity of superspace coordinates naturally appears on a D-brane worldvolume, this time due to the presence of Ramond-Ramond 2-form field strength in ten dimensions or equivalently, a self-dual graviphoton field strength in four dimensions. Interestingly, the bosonic coordinates still commute, but the fermionic superspace coordinates do not anticommutate and instead, satisfy a clifford algebra $\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}$ [14]–[23]. Recently it was shown in [13], that such theories with $\mathcal{N} = 1$ supersymmetry in $D = 4$

are consistent and Lorentz invariant at the classical level. A surprising aspect is that the non(anti)commutativity parameter C appears explicitly in the superalgebra, and breaks half of the supersymmetry. Consequently, the superspace itself becomes half supersymmetric.

The ensuing studies [24]–[45], have explored the idea of non(anti)commutativity of superspace coordinates, in connection with matrix models [27, 29, 35], soliton and instanton solutions [41, 42], UV/IR mixing [31] etc.. At the quantum level, sufficient work has been devoted to the perturbative study of supersymmetric gauge theories and Wess-Zumino model with $\mathcal{N} = 1$ supersymmetry in four dimensions [26, 37, 40, 43]. Regarding these models, certain (non)renormalization theorems have been proved [31] and also renormalizability [32]–[34] to all orders in perturbation theory has been shown. Furthermore, four dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry have also been formulated on a non(anti)commutative superspace [38, 39].

Although, much of the focus has been in connection with the $\mathcal{N} = 1, 2$ supersymmetric models in four space-time dimensions, it is both interesting and important to explore the effects of non(anti)commutativity in more general contexts. Especially, $D = 2$ offers an interesting arena where nonlinear σ-models have been studied in great detail. The relation of ultraviolet structure of Bosonic σ-models to the target manifold on which they are defined has also been explored [46]. Recently, noncommutative nonlinear σ-models have also been studied in considerable detail [47]–[51].

It is well known that supersymmetric generalization of Bosonic σ-models, unveiled deep connections between the complex manifold theory and supersymmetry [52]–[62]. The restriction on the type of manifolds compatible with extended supersymmetry has compelled the study of Kähler manifolds, besides leading to the construction of new, previously unknown manifolds [59]. To be precise, in connection with the supersymmetric extension of Bosonic σ-models, it was shown by Zumino [52], and by Alvarez-Gaumé and Freedman [53]–[56], that $\mathcal{N} = 1$ supersymmetry is possible on any arbitrary manifold, but $\mathcal{N} = 2$ supersymmetry requires the manifold to be Kähler and $\mathcal{N} = 4$ supersymmetry can only be realized if the manifold is hyperKähler. Remarkably, Kähler geometry has proved to have strong implications for the renormalizability of $\mathcal{N} = 2$ supersymmetric σ-models defined on either riemannian or Kähler manifolds [54]–[56].

Also, the importance of $\mathcal{N} = 2$ supersymmetric sigma model in the context of $\mathcal{N} = 2$ strings is well known [63]–[66]. In the recent past, noncommutative $\mathcal{N} = 2$ strings have also been discussed [67, 68]. Motivated by the above results, in this paper, we explore the implications of Kähler structure for two dimensional $\mathcal{N} = 2$ supersymmetric theories on a non(anti)commutative superspace.

The rest of the paper is organized as follows. In section 2, we present our $\mathcal{N} = 2$ ($D = 2$) non(anti)commutative superspace, while also reviewing certain general properties of Kähler manifolds. In section 3, we obtain the classical action for an $\mathcal{N} = 2$ supersymmetric theory defined by an arbitrary Kähler potential of a single chiral and antichiral superfield and show the emergence of a power series expansion in (det C). In section 4, we discuss the supersymmetry properties of our theory and generalize the results to include the superpotential. Conclusions and discussion are presented in section 5.

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2. Non(anti)commutative superspace

We start by formulating our $\mathcal{N}=2$ supersymmetric theory, on a non(anti)commutative superspace. In four dimensions, it was noted [17], that the algebra of superspace coordinates can be made non(anti)commutative and also associative only in a Euclidean space-time. Further in [13], it was elucidated that starting from an $\mathcal{N}=1$ supersymmetric theory, non(anti)commutative algebra can be consistently formulated on a euclidean space. Coming to two dimensions, there is a well known connection between $D=4, \mathcal{N}=1$ and $D=2, \mathcal{N}=2$ models, suggesting the existence of an associative non(anti)commutative algebra, as also pointed out in [44]. We now define this superspace in $D=2$ and continue to use lorentzian signature as in [13].

2.1 $D=2$, $\mathcal{N}=2$ superspace

We begin establishing our notations [63, 69]. The superspace coordinates are denoted by: $\tau, \sigma, \theta^i_\alpha$ [63]. Here, τ, σ are space-time labels and θ^i_α are N Majorana spinors which are odd elements of a Grassmann algebra i.e., $\theta^i_\alpha \theta^i_\beta = -\theta^i_\beta \theta^i_\alpha$ [63]. For later convenience, we define the light-cone coordinates:

$$\xi = \frac{1}{2}(\tau+\sigma)\,, \qquad \zeta = \frac{1}{2}(\tau-\sigma)\,, \tag{2.1}$$

and for spinor coordinates, one can invoke the chirality condition using γ_5 in two space-time dimensions [63]:

$$\theta^i = \frac{1}{2}(1-\gamma_5)_{\alpha\beta}\theta^i_\beta, \qquad \chi^i = \frac{1}{2}(1+\gamma_5)_{\alpha\beta}\theta^i_\beta\,, \tag{2.2}$$

with [63]:

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

In what follows, we will be using the complex version of the spinors defined below, which can all be taken to be independent [44]:

$$\theta = \theta^1 + i\theta^2\,, \qquad \bar\theta = \theta^1 - i\theta^2\,, \qquad \chi = \chi^1 + i\chi^2\,, \qquad \bar\chi = \chi^1 - i\chi^2\,. \tag{2.4}$$

Hence, using the known results for $D=4$, $\mathcal{N}=1$ [13], and $D=2$, $\mathcal{N}=2$ [44], the non(anti)commutativity between the grassmannian coordinates (2.4), can be introduced as shown below:

$$\{\theta,\theta\} = C^{00}\,, \qquad \{\theta,\chi\} = -C^{01}\,, \qquad \{\chi,\theta\} = -C^{10}\,, \qquad \{\chi,\chi\} = C^{11}\,, \tag{2.5}$$

with all other anticommutators of $\theta, \chi, \bar\theta, \bar\chi$ vanishing. Notice that from eqn. (2.5), we have $C^{01} = C^{10}$ which will be used later. Also,

$$\begin{aligned} [\bar\theta,\xi] &= [\bar\chi,\xi] = 0\,, \\ [\bar\theta,\zeta] &= [\bar\chi,\zeta] = 0\,. \end{aligned} \tag{2.6}$$

An immediate consequence of (2.5) is that, when functions of non(anti)commutative coordinates i.e., θ, χ are multiplied, the result has to be Weyl ordered. This can be systematically implemented, by introducing a star product between the non(anti)commutative coordinates, as defined below [13]:

$$f(\theta,\chi) * g(\theta,\chi) \;=\; f(\theta,\chi)\exp\left(-\frac{C^{00}}{2}\overleftarrow{\partial_\theta}\overrightarrow{\partial_\theta} + \frac{C^{01}}{2}\overleftarrow{\partial_\theta}\overrightarrow{\partial_\chi} + \frac{C^{10}}{2}\overleftarrow{\partial_\chi}\overrightarrow{\partial_\theta} - \frac{C^{11}}{2}\overleftarrow{\partial_\chi}\overrightarrow{\partial_\chi}\right) g(\theta,\chi). \tag{2.7}$$

$$= f(\theta,\chi)\Big[1 - \frac{C^{00}}{2}\overleftarrow{\partial_\theta}\overrightarrow{\partial_\theta} + \frac{C^{01}}{2}\overleftarrow{\partial_\theta}\overrightarrow{\partial_\chi} + \frac{C^{10}}{2}\overleftarrow{\partial_\chi}\overrightarrow{\partial_\theta} - \frac{C^{11}}{2}\overleftarrow{\partial_\chi}\overrightarrow{\partial_\chi} - \tag{2.8}$$
$$- \left(\frac{1}{16}\det C\right)(\overleftarrow{\partial_\theta}\overleftarrow{\partial_\chi} - \overleftarrow{\partial_\chi}\overleftarrow{\partial_\theta})(\overrightarrow{\partial_\theta}\overrightarrow{\partial_\chi} - \overrightarrow{\partial_\chi}\overrightarrow{\partial_\theta})\Big] g(\theta,\chi)\,.$$

Coming to the algebra of bosonic coordinates (2.1), it has been noted that if we assume,

$$[\xi,\xi] = [\zeta,\zeta] = [\xi,\zeta] = 0\,, \tag{2.9}$$

then the supersymmetry remains unbroken, but the supercovariant derivatives do not act as derivations and hence, defining antichiral fields becomes difficult. Alternatively, as pointed out in [13], to define antichiral superfields one works in the chiral coordinate basis defined as:

$$\xi^{\pm} = \xi \pm \frac{i}{2}\theta\bar{\theta}\,, \qquad \zeta^{\pm} = \zeta \pm \frac{i}{2}\chi\bar{\chi}\,. \tag{2.10}$$

To be consistent with the choice in eqns. (2.5) and (2.6), we impose the following relations:

$$\begin{aligned}
[\xi^-,\xi^-] &= [\zeta^-,\zeta^-] = [\xi^-,\zeta^-] = 0\,,\\
[\xi^-,\theta] &= [\xi^-,\bar{\theta}] = [\xi^-,\chi] = [\xi^-,\bar{\chi}] = 0\,,\\
[\zeta^-,\theta] &= [\zeta^-,\bar{\theta}] = [\zeta^-,\chi] = [\zeta^-,\bar{\chi}] = 0\,.
\end{aligned} \tag{2.11}$$

As a consequence, we get the following identities:

$$\begin{aligned}
[\xi,\xi] &= 0 = [\zeta,\zeta]\,, & [\xi,\zeta] &= \frac{1}{4}\,\bar{\theta}\bar{\chi}C^{01}\,,\\
[\xi,\theta] &= -\frac{i}{2}\bar{\theta}C^{00}\,, & [\zeta,\chi] &= -\frac{i}{2}\bar{\chi}C^{11}\,,\\
[\xi,\chi] &= \frac{i}{2}\bar{\theta}C^{01}\,, & [\zeta,\theta] &= \frac{i}{2}\bar{\chi}C^{10}\,.
\end{aligned} \tag{2.12}$$

Now, we can write down the supercovariant derivatives in the chiral basis as:

$$D_1 = -\frac{\partial}{\partial\theta} + i\bar{\theta}\frac{\partial}{\partial\xi^-}\,, \qquad \bar{D}_1 = -\frac{\partial}{\partial\bar{\theta}}\,, \tag{2.13}$$

$$D_2 = \frac{\partial}{\partial\chi} - i\bar{\chi}\frac{\partial}{\partial\zeta^-}\,, \qquad \bar{D}_2 = \frac{\partial}{\partial\bar{\chi}}\,, \tag{2.14}$$

and the supercharges are given as:

$$Q_1 = -\frac{\partial}{\partial\theta}\,, \qquad \bar{Q}_1 = -\frac{\partial}{\partial\bar{\theta}} - i\theta\frac{\partial}{\partial\xi^-}\,, \tag{2.15}$$

$$Q_2 = \frac{\partial}{\partial\chi}\,, \qquad \bar{Q}_2 = \frac{\partial}{\partial\bar{\chi}} + i\chi\frac{\partial}{\partial\zeta^-}\,. \tag{2.16}$$

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We can now write down the algebra of supercovariant derivatives using eqns. (2.13) and (2.14) as:

$$\begin{aligned}
\{D_1, D_1\} &= \{D_2, D_2\} = \{D_1, \bar{D}_2\} = \{D_2, \bar{D}_1\} = 0 && (2.17)\\
\{\bar{D}_1, \bar{D}_1\} &= \{\bar{D}_2, \bar{D}_2\} = 0 && (2.18)\\
\{D_1, \bar{D}_1\} &= -i\partial_{\xi^-}\,, \qquad \{D_2, \bar{D}_2\} = -i\partial_{\zeta^-}\,. && (2.19)
\end{aligned}$$

The above algebra turns out to be same as in the usual superspace. In addition, all the supercovariant derivatives anticommute with all the supercharges. Also, using the definitions given in eqns. (2.15) and (2.16) the supercharges can be shown to satisfy:

$$\begin{aligned}
&\{Q_1, Q_1\} = \{Q_2, Q_2\} = \{Q_1, \bar{Q}_2\} = \{Q_2, \bar{Q}_1\} = 0, && (2.20)\\
&\{Q_1, \bar{Q}_1\} = i\partial_{\xi^-}, \qquad \{Q_2, \bar{Q}_2\} = i\partial_{\zeta^-}, && (2.21)\\
&\{\bar{Q}_1, \bar{Q}_1\} = -C^{00}\frac{\partial^2}{\partial_{\xi^-}\partial_{\zeta^-}}, \qquad \{\bar{Q}_2, \bar{Q}_2\} = -C^{11}\frac{\partial^2}{\partial_{\xi^-}\partial_{\zeta^-}}. && (2.22)
\end{aligned}$$

An important aspect of the above algebra, as first noted in [13] is that, $\bar{Q}$ is no more a symmetry on the non(anti)commutative space, whereas Q still continues to be the symmetry. Since, half of the supersymetry is broken, in our case, the unbroken Q supersymmetry can be termed as $N = 2/2$ supersymmetry.

Before proceeding, we write down certain identities derived from the star product (2.7) of grassmannian coordinates given in eqn. (2.4), for later use:

$$\begin{aligned}
\theta * \theta &= \theta\theta + \frac{1}{2}C^{00}\,, \qquad \chi * \chi = \chi\chi + \frac{1}{2}C^{11}\,, && (2.23)\\
\theta * \chi &= (\theta\chi) - \frac{1}{2}C^{01}\,, \qquad \chi * \theta = -(\theta\chi) - \frac{1}{2}C^{10}\,, && (2.24)\\
\theta * \Big(\theta\chi\Big) &= -\theta * \Big(\chi\theta\Big) = \frac{1}{2}\Big(C^{00}\chi + C^{01}\theta\Big)\,, && (2.25)\\
\chi * \Big(\chi\theta\Big) &= -\chi * \Big(\theta\chi\Big) = \frac{1}{2}\Big(C^{11}\theta + C^{10}\chi\Big)\,, && (2.26)\\
\Big(\theta\chi\Big) * \Big(\theta\chi\Big) &= -\Big(\chi\theta\Big) * \Big(\theta\chi\Big) = -\frac{1}{4}\,(\det\, C)\,. && (2.27)
\end{aligned}$$

To construct a superspace action, we need the chiral and antichiral superfields, which we define below. Chiral superfields satisfy $\bar{D}_1 S = 0 = \bar{D}_2 S$ and can be written as:

$$S(\xi^-, \zeta^-, \theta, \chi) = A(\xi^-, \zeta^-) + i\theta\bar{\psi}_L(\xi^-, \zeta^-) - i\chi\bar{\psi}_R(\xi^-, \zeta^-) - \frac{1}{2}i\Big(\theta * \chi - \chi * \theta\Big)F(\xi^-, \zeta^-)\,, \tag{2.28}$$

where the star product of θ and χ appearing in eqn. (2.28) can be shown to be Weyl ordered after using $C^{01} = C^{10}$, as shown below:

$$\Big(\theta * \chi - \chi * \theta\Big) = \Big(\theta\chi - \chi\theta\Big) = 2(\theta\chi)\,. \tag{2.29}$$

Now, antichiral superfields satisfying $D_1\bar{S} = 0 = D_2\bar{S}$ can be written as:

$$\bar{S}(\xi^+, \zeta^+, \bar{\theta}, \bar{\chi}) = \bar{A}(\xi^+, \zeta^+) - i\bar{\theta}\psi_L(\xi^+, \zeta^+) + i\bar{\chi}\psi_R(\xi^+, \zeta^+) - i\bar{\theta}\bar{\chi}\bar{F}(\xi^+, \zeta^+)\,. \tag{2.30}$$

It has been pointed out in [13] that, since (ξ^+, ζ^+) do not commute among themselves, one needs to Weyl order functions of these coordinates as well, which is inconvenient. Hence, using the definitions given in (2.10), we expand the antichiral superfield around (ξ^-, ζ^-) coordinates, so that, we only Weyl order the θ's. Doing so, one ends up with [13]:

$$\begin{aligned}\bar{S}(\xi^+, \zeta^+, \bar{\theta}, \bar{\chi}) &= \bar{A}(\xi^-, \zeta^-) - i\bar{\theta}\psi_L(\xi^-, \zeta^-) + i\bar{\chi}\psi_R(\xi^-, \zeta^-) + i\theta\bar{\theta}\partial_{\xi^-}\bar{A} + i\chi\bar{\chi}\partial_{\zeta^-}\bar{A} + \\ &\quad + \bar{\theta}\bar{\chi}\left(-\chi\partial_{\zeta^-}\psi_L - \theta\partial_{\xi^-}\psi_R - i\bar{F} + (\theta\chi)\partial_{\xi^-}\partial_{\zeta^-}\bar{A}\right). \end{aligned} \tag{2.31}$$

Further, the star product does not break chirality. The star product of two chiral superfields is still chiral, as seen below:

$$S(\xi^-, \zeta^-, \theta, \chi) * S(\xi^-, \zeta^-, \theta, \chi) = S(\xi^-, \zeta^-, \theta, \chi) S(\xi^-, \zeta^-, \theta, \chi) + \left(\frac{1}{4}\det C\right) F^2 . \tag{2.32}$$

Besides, it is interesting to note the appearance of the non(anti)commutativity parameter as $(\det C)$. One can also check that the star product does not break antichirality as well, i.e., the star product of two antichiral superfields still remains antichiral.

2.2 σ-models and Kähler geometry

Here, we give a brief review of the known results concerning the supersymmetric extension of the Bosonic nonlinear σ-models and also write identities coming from the structure of the Kähler geometry [70, 71].

Let us start by writing down the action for the bosonic nonlinear σ-model on a general even dimensional riemannian manifold M, with a real metric $g_{i\bar{j}}(z, \bar{z})$ and the complex coordinate fields $z^i, \bar{z}^j$, where $i, j = 1, \ldots, n$, as shown below:

$$I = \int d^2x \; g_{i\bar{j}} \, \partial_\mu z^i \partial_\mu \bar{z}^j . \tag{2.33}$$

Constraining the manifold to be hermitean, the unmixed components of the metric vanish, i.e., $g_{ij} = g_{\bar{i}\bar{j}} = 0$. Moreover, as has been already said, to couple the bosonic σ-model (2.33) to n complex spinor fields and have $\mathcal{N} = 2$ supersymmetric extension, the target manifold has to be Kähler.

The line element on a Kähler manifold can be written locally as:

$$ds^2 = 2g_{i\bar{j}} \, dz^i d\bar{z}^j , \tag{2.34}$$

where the hermitean metric $g_{i\bar{j}}$ can be obtained (locally) as a second derivative (once holomorphic and once anti-holomorphic) of an arbitrary real scalar function (Kähler potential), say, $\mathcal{K}(z, \bar{z})$ as shown:

$$g_{i\bar{j}} = \frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^j}\mathcal{K}(z, \bar{z}) . \tag{2.35}$$

The above definition of metric (2.35), is invariant under Kähler gauge transformations and arbitrary holomorphic coordinate transformations:

$$\begin{aligned}\mathcal{K}'(z, \bar{z}) &= \mathcal{K}(z, \bar{z}) + \Lambda(z) + \bar{\Lambda}(\bar{z}) , \\ (z^i)' &= z'^i(z^j) , \qquad (\bar{z}^i)' = \bar{z}'^i(\bar{z}^j) . \end{aligned} \tag{2.36}$$

Evidently the metric (2.35) is real and off diagonal with the only nonzero components being $g_{i\bar{j}}$ and $g_{\bar{j}i}$. The non vanishing components of the Kähler metric and their inverses, further satisfy:

$$g_{i\bar{j}}\, g^{k\bar{j}} = \delta_i^k\,, \qquad g^{i\bar{j}}\, g_{i\bar{k}} = \delta_{\bar{k}}^{\bar{j}}\,. \tag{2.37}$$

From the Kähler condition,

$$\partial_k g_{i\bar{j}}(z,\bar{z}) = \partial_i g_{k\bar{j}}(z,\bar{z})\,, \tag{2.38}$$

many simplifications occur and very few components of the Christoffel symbols and the Riemann Curvature tensor remain nonzero. We first write down the non vanishing components of Christoffel symbols and the Riemann Curvature tensor in terms of the Kähler potential $\mathcal{K}(z,\bar{z})$:

$$\begin{aligned}
\Gamma^i_{jk} &= g^{i\bar{l}}\, \frac{\partial}{\partial z^j}\frac{\partial}{\partial z^k}\frac{\partial}{\partial \bar{z}^l}\, \mathcal{K}(z,\bar{z})\,, \qquad \Gamma^{\bar{i}}_{\bar{j}\bar{k}} = g^{\bar{i}l}\, \frac{\partial}{\partial \bar{z}^j}\frac{\partial}{\partial \bar{z}^k}\frac{\partial}{\partial z^l}\, \mathcal{K}(z,\bar{z})\,, \\
R^i_{j\bar{k}l} &= g^{i\bar{m}}\, \frac{\partial}{\partial z^j}\frac{\partial}{\partial z^l}\frac{\partial}{\partial \bar{z}^k}\frac{\partial}{\partial \bar{z}^m}\, \mathcal{K}(z,\bar{z})\,.
\end{aligned} \tag{2.39}$$

Starting from the basic definition given in eqn. (2.35), one can also deduce the following identities:

$$\begin{aligned}
\Gamma^i_{jk} &= g^{i\bar{l}}\, \partial_j g_{k\bar{l}}\,, \qquad & \Gamma^{\bar{i}}_{\bar{j}\bar{k}} &= g^{\bar{i}l}\, \partial_{\bar{j}} g_{\bar{k}l}\,, \qquad & \Gamma_{\bar{i}jk} &= g_{\bar{i}l}\,\Gamma^l_{jk}\,, \qquad & \Gamma_{i\bar{j}\bar{k}} &= g_{i\bar{l}}\,\Gamma^{\bar{l}}_{\bar{j}\bar{k}}\,, \\
R^i_{j\bar{k}l} &= \partial_{\bar{k}}\Gamma^i_{jl}\,, & R_{\bar{i}j\bar{k}l} &= g_{\bar{i}m} R^m_{j\bar{k}l}\,, & R_{\bar{i}j\bar{k}l} &= R_{\bar{i}l\bar{k}j}\,.
\end{aligned} \tag{2.40}$$

Furthermore, the Ricci tensor $R_{i\bar{j}} = g^{m\bar{n}}\, R_{\bar{n}i\bar{j}m}$, is a "Kähler tensor" as it satisfies $R_{ij} = R_{\bar{i}\bar{j}} = 0$ and locally it can be obtained as $R_{i\bar{j}} = \partial_i\partial_{\bar{j}} \ln\det(g^{k\bar{l}})$.

3. $\mathcal{N}=2$ supersymmetric theory on non(anti)commutative superspace

In this section, we derive the classical action for $\mathcal{N}=2$ supersymmetric theories with Kähler structure on non(anti)commutative superspace and then in section 4, show that half of the supersymmetry remains unbroken. In this paper, we restrict ourselves to a single chiral and antichiral supermultiplet. The generalization to include n chiral and antichiral superfields remains a subject of future work.

3.1 Classical action

Continuing the discussion in section 2.2, the most general action for $\mathcal{N}=2$ supersymmetric theories characterized by a Kähler potential $\mathcal{K}(S,\bar{S})$, where S and $\bar{S}$ are chiral and antichiral superfields, takes an extremely simple form as shown below:

$$I = \int d^2x\, d\theta\, d\chi\, d\bar{\theta}\, d\bar{\chi}\, \mathcal{K}(S,\bar{S})\,, \tag{3.1}$$

where S and $\bar{S}$ are chiral and antichiral superfields defined in eqns. (2.28) and (2.31). To obtain the action in terms of the component fields, one expands $\mathcal{K}(S,\bar{S})$ around the bosonic fields, A and $\bar{A}$. Following this general procedure in our case and using the definitions of chiral and antichiral superfields given in eqns. (2.28) and (2.31), we first explicitly write down the possible terms in the expansion of the Kähler potential $\mathcal{K}(S,\bar{S})$:

$$
\begin{aligned}
\mathcal{K}(S,\bar{S}) \,=\, & \mathcal{K}(A,\bar{A}) + L\frac{\partial\mathcal{K}}{\partial S} + R\frac{\partial\mathcal{K}}{\partial\bar{S}} + \frac{1}{2!}L*L\frac{\partial^2\mathcal{K}}{\partial S^2} + \frac{1}{2!}R*R\frac{\partial^2\mathcal{K}}{\partial\bar{S}^2} + \\
& + \frac{1}{2!}\left[L*R\right]\frac{\partial^2\mathcal{K}}{\partial S\partial\bar{S}} + \frac{1}{3!}\left[L_*^2*R\right]\frac{\partial^3\mathcal{K}}{\partial S^2\partial\bar{S}} + \frac{1}{3!}\left[R_*^2*L\right]\frac{\partial^3\mathcal{K}}{\partial S\partial\bar{S}^2} + \\
& + \cdots + \frac{1}{n!}\,L_*^n\,\frac{\partial^n\mathcal{K}}{\partial S^n} + \cdots + \frac{1}{n!}\,R_*^n\,\frac{\partial^n\mathcal{K}}{\partial\bar{S}^n} + \cdots + \\
& + \frac{1}{(n+m)!}\left[L_*^n * R_*^m\right]\frac{\partial^{n+m}\mathcal{K}}{\partial S^n\partial\bar{S}^m} + \cdots\,, \qquad (3.2)
\end{aligned}
$$

where n and m are integers and we have also introduced the following notations:

$$
L = S - A, \qquad R = \bar{S} - \bar{A}\,, \qquad (3.3)
$$

$$
L_*^n = \overbrace{L*L*\cdots*L}^{n}\,, \qquad R_*^m = \overbrace{R*R*\cdots*R}^{m}\,, \qquad (3.4)
$$

and the square brackets $[\cdots]$ in eqn. (3.2), signify all possible combinations of L's and R's. For instance, in this notation: $[L*L*R] \equiv L*L*R + L*R*L + R*L*L$.

Also, to keep the notations simple in eqn. (3.2),

$$
\frac{\partial\mathcal{K}}{\partial S}, \frac{\partial^2\mathcal{K}}{\partial S^2}, \ldots, \frac{\partial^{n+m}\mathcal{K}}{\partial S^n\partial\bar{S}^m} \quad \text{etc.}, \qquad (3.5)
$$

stand for the derivatives of the Kähler potential, evaluated at $S = A$ and $\bar{S} = \bar{A}$.

For L and R, explicitly one has:

$$
\begin{aligned}
L(\xi^-,\zeta^-,\theta,\chi) \,&=\, +i\theta\,\bar{\psi}_L - i\chi\,\bar{\psi}_R - i(\theta\,\chi)\,F\,, \qquad (3.6)\\
R(\xi^+,\zeta^+,\bar{\theta},\bar{\chi}) \,&=\, -i\bar{\theta}\,\psi_L + i\bar{\chi}\psi_R + i\theta\bar{\theta}\partial_{\xi^-}\bar{A} + i\chi\bar{\chi}\partial_{\zeta^-}\bar{A} + \\
&\quad + \bar{\theta}\,\bar{\chi}\left(-\chi\,\partial_{\zeta^-}\psi_L - \theta\,\partial_{\xi^-}\psi_R - i\bar{F} + (\theta\,\chi)\,\partial_{\xi^-}\partial_{\zeta^-}\bar{A}\right), \qquad (3.7)
\end{aligned}
$$

where we have suppressed the dependence of component fields on the coordinates (ξ^-,ζ^-). All the component fields are taken to be functions of (ξ^-,ζ^-), if not explicitly mentioned. Using the form of L and R given in eqns. (3.6) and (3.7), we arrive at the following identities:

$$
\begin{aligned}
L_*^{2n} \,&=\, \left(\frac{1}{4}\det C\right)^{n-1} F^{2n-2}\left[-2n(\theta\chi)\bar{\psi}_L\,\bar{\psi}_R + \left(\frac{1}{4}\det C\right)F^2\right], \qquad (3.8)\\
L_*^{2n+1} \,&=\, \left(\frac{1}{4}\det C\right)^{n} F^{2n-1}\left(-2n\,i\,\bar{\psi}_L\,\bar{\psi}_R + \text{F.L.}\right), \qquad (3.9)\\
R_*^2 \,&=\, 2\bar{\theta}\bar{\chi}\Big(-\psi_L\psi_R - \chi\,\psi_L\,\partial_{\zeta^-}\bar{A} - \theta\,\psi_R\,\partial_{\xi^-}\bar{A} + (\theta\,\chi)\partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A}\Big), \qquad (3.10)\\
R_*^n \,&=\, 0\,, \quad \text{for} \quad n > 2\,. \qquad (3.11)
\end{aligned}
$$

To evaluate the action in eqn. (3.1) explicitly, one needs to collect the terms with coefficient $\bar{\theta}\bar{\chi}(\theta\chi)$ appearing in the expansion of Kähler potential in eqn. (3.2), as only these terms survive after performing integration over the Grassmann variables. Now, from eqns. (3.6), (3.8) and (3.9), one can see that the star product of an arbitrary number of

L's alone does not have a term proportional to $\bar\theta\,\bar\chi(\theta\,\chi)$, and hence, it cannot give rise to terms in the action. On the other hand, with the same logic, R given in eqn. (3.7) the star product of two R's given in eqn. (3.10), do contribute terms to the action. Further, the star product of three and higher number of R's vanishes and hence, does not contribute to the action. Therefore, to proceed, we turn towards the star product of an arbitrary number of L's with either R or $R*R$. First, using the definitions of L and R given in eqns. (3.6) and (3.7), we arrive at the following relation:

$$\begin{aligned}
L*R+R*L = &-2\bar\chi\theta\bar\psi_L\psi_R - 2\bar\theta\chi\bar\psi_R\psi_L + 2\bar\theta\theta\bar\psi_L\psi_L + 2\bar\chi\chi\bar\psi_R\psi_R + \\
&+\frac{i}{2}\bar\theta\bar\chi\,(\det C)\,F\partial_{\xi^-}\partial_{\zeta^-}\bar A - 2\bar\theta(\theta\chi)\left(\bar\psi_R\partial_{\xi^-}\bar A + F\psi_L\right) - \\
&-2\bar\chi(\theta\chi)\left(\bar\psi_L\,\partial_{\zeta^-}\bar A - F\psi_R\right) + 2\bar\theta\bar\chi\theta\bar\psi_L\bar F - 2\bar\theta\bar\chi\chi\bar\psi_R\bar F + \\
&+2\bar\theta\bar\chi(\theta\chi)\left(i\bar\psi_L\partial_{\zeta^-}\psi_L + i\bar\psi_R\,\partial_{\xi^-}\psi_R - F\,\bar F\right). \qquad (3.12)
\end{aligned}$$

Note that in the above equation, $L*R$ and $R*L$ independently have many more terms (explicit expressions are given in the appendix) which cancel out when we consider $[L*R]$. Further, since our interest is in extracting only the terms with coefficient $\bar\theta\bar\chi(\theta\chi)$, we mainly concentrate on those terms. Moreover, in this connection, we make the observation that, focusing only on the terms with coefficient $\bar\theta\bar\chi(\theta\chi)$, a great deal of simplification occurs, allowing us to obtain several identities which we give in the appendix. The crux of the matter is that a general term in the expansion in eqn. (3.2), e.g., $[L_*^p*R*L_*^q*R*L_*^r]$, can give rise to various terms in the action corresponding to all possible combinations of L's and R's. However, using the identities given in the appendix, one can push all the L's to one side and all the R's to the other side as:

$$[L_*^p*R*L_*^q*R*L_*^r]\,|_{\bar\theta\bar\chi\theta\chi} \;\equiv\; {}^{(p+q+r+2)}C_2 L_*^{(p+q+r)} * R_*^2\,|_{\bar\theta\bar\chi\theta\chi}\,, \qquad (3.13)$$

where p, q, r are integers and in writing the combinatorial factor, we have used the notation $n!/k!(n-k)! = {}^{(n)}C_k$. We give the proof of these identities in the appendix and in what follows use these results directly. Explicit check of supersymmetry (in section 4) further confirms that our action is indeed correct.

The star product of even and odd number of L's with R, can be shown to satisfy (when restricted to terms with coefficient $\bar\theta\bar\chi(\theta\chi)$):

$$\begin{aligned}
\left[L_*^{2n}*R\right]|_{\bar\theta\bar\chi\theta\chi} &= {}^{(2n+1)}C_1\,L_*^{2n}*R\,|_{\bar\theta\bar\chi\theta\chi} \qquad (3.14)\\
&= (2n+1)\left(\frac{1}{4}\det C\right)^{n-1} F^{2n-2}\left[2ni\bar\psi_L\bar\psi_R\bar F + \left(\frac{1}{4}\det C\right)F^2\partial_{\xi^-}\partial_{\zeta^-}\bar A\right],\\
\left[L_*^{2n+1}*R\right]|_{\bar\theta\bar\chi\theta\chi} &= {}^{(2n+2)}C_1 L_*^{2n+1}*R|_{\bar\theta\bar\chi\theta\chi}\\
&= (2n+2)\left(\frac{1}{4}\det C\right)^{n} F^{2n-1}\times\\
&\quad\times\left[-2ni\bar\psi_L\bar\psi_R\partial_{\xi^-}\partial_{\zeta^-}\bar A + F\left(i\bar\psi_L\partial_{\zeta^-}\psi_L + i\bar\psi_R\partial_{\xi^-}\psi_R - F\bar F\right)\right], \qquad (3.15)
\end{aligned}$$

where $n \geq 1$. Now, one can notice that eqn. (3.12), has terms proportional to $\bar{\theta}\bar{\chi}(\theta\chi)$ which are also independent of C. Therefore, these terms would give rise to the standard $C = 0$, $\mathcal{N} = 2$ supersymmetric action. On the other hand, from the set of eqns. (3.14) and (3.15), and similar ones appearing below, one can see the emergence of new terms in the action, proportional to arbitrary powers of $(\det C)$.

Next, we go on to compute the star product of arbitrary powers of L's with R_*^2, needed for writing down our action. Using the results given for L's in eqns. (3.8) and (3.9) and R_*^2 in (3.10), we end up with:

$$\begin{aligned}
\left[L * R_*^2\right]|_{\bar{\theta}\bar{\chi}\theta\chi} &= {}^3C_1 L * R_*^2|_{\bar{\theta}\bar{\chi}\theta\chi} \\
&= 6i\left(\psi_L\psi_R F + \bar{\psi}_L\psi_L\partial_{\zeta^-}\bar{A} + \bar{\psi}_R\psi_R\partial_{\xi^-}\bar{A}\right), \qquad (3.16) \\
\left[L_*^{2n} * R_*^2\right]|_{\bar{\theta}\bar{\chi}\theta\chi} &= {}^{(2n+2)}C_2 L_*^{2n} * R_*^2|_{\bar{\theta}\bar{\chi}\theta\chi} \\
&= {}^{(2n+2)}C_2 2\left(\frac{1}{4}\det C\right)^{n-1} F^{2n-2} \times \\
&\quad \times \left[2n\bar{\psi}_L\bar{\psi}_R\psi_L\psi_R + \left(\frac{1}{4}\det C\right)F^2\partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A}\right], \qquad (3.17) \\
\left[L_*^{2n+1} * R_*^2\right]|_{\bar{\theta}\bar{\chi}\theta\chi} &= {}^{(2n+3)}C_2 L_*^{2n+1} * R_*^2|_{\bar{\theta}\bar{\chi}\theta\chi} \\
&= -{}^{(2n+3)}C_2 2i\left(\frac{1}{4}\det C\right)^{n} F^{2n-1} \times \qquad (3.18) \\
&\quad \times \left[2n\bar{\psi}_L\bar{\psi}_R\partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A} - F\left(\bar{\psi}_L\psi_L\partial_{\zeta^-}\bar{A} + \bar{\psi}_R\psi_R\partial_{\xi^-}\bar{A} + \psi_L\psi_R F\right)\right],
\end{aligned}$$

where $n \geq 1$. It may be noted that eqn. (3.16) contributes terms in the standard $C = 0$, supersymmetric $\mathcal{N} = 2$ action. Furthermore, eqns. (3.17) and (3.18) contribute new terms in the action, proportional to various powers of $(\det C)$.

To summarize, substituting the results obtained in eqns. (3.6)-(3.18) in the action (3.1), we end up with the classical action for supersymmetric $\mathcal{N} = 2$ theory on a non(anti)commutative superspace. It is possible to break the action into two parts as:

$$I = I_0 + I_C^n, \qquad (3.19)$$

where I_0 corresponds to the $C = 0$, $\mathcal{N} = 2$ action and I_C^n stands for the non(anti)commutative part of the $\mathcal{N} = 2$ theory.

Below, we first collect the terms coming from the expansion of the Kähler potential in the action in eqn. (3.1), which will ultimately give rise to I_0:

$$\begin{aligned}
I_0 = \int d^2x d\theta d\chi d\bar{\theta} d\bar{\chi}\Big[&R\frac{\partial\mathcal{K}}{\partial\bar{S}} + \frac{1}{2!}R_*^2\frac{\partial^2\mathcal{K}}{\partial\bar{S}^2} + \frac{1}{2!}\left[L * R\right]\frac{\partial^2\mathcal{K}}{\partial S\partial\bar{S}} + \frac{1}{3!}\left[L_*^2 * R\right]\frac{\partial^3\mathcal{K}}{\partial S^2\partial\bar{S}} + \\
&+ \frac{1}{3!}\left[R_*^2 * L\right]\frac{\partial^3\mathcal{K}}{\partial S\partial\bar{S}^2} + \frac{1}{4!}\left[L_*^2 * R_*^2\right]\frac{\partial^4\mathcal{K}}{\partial S^2\partial\bar{S}^2}\Big]. \qquad (3.20)
\end{aligned}$$

Now, making use of the identities derived in eqns. (3.7), (3.10), (3.12), (3.14), (3.16) and (3.17), in the above form of the action and after performing integration over the grassmannian coordinates, in the usual way, we end up with the following action for $C = 0$,

$\mathcal{N}=2$ supersymmetric theory:

$$
\begin{aligned}
I_0 = \int d^2x \Big[& \Big(\partial_{\xi^-}\partial_{\zeta^-}\bar{A}\Big)\frac{\partial\mathcal{K}}{\partial\bar{S}} + \Big(i\bar{\psi}_L\partial_{\zeta^-}\psi_L + i\bar{\psi}_R\partial_{\xi^-}\psi_R - F\bar{F}\Big)\frac{\partial^2\mathcal{K}}{\partial S\partial\bar{S}} + \Big(\partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A}\Big)\times \\
& \times \frac{\partial^2\mathcal{K}}{\partial\bar{S}^2} + i\,(\,\bar{\psi}_L\bar{\psi}_R\bar{F})\,\frac{\partial^3\mathcal{K}}{\partial S^2\partial\bar{S}} + \\
& + i\,(\psi_L\psi_R F + \bar{\psi}_L\psi_L\partial_{\zeta^-}\bar{A} + \bar{\psi}_R\psi_R\partial_{\xi^-}\bar{A})\,\frac{\partial^3\mathcal{K}}{\partial S\partial\bar{S}^2} - (\bar{\psi}_L\psi_L\bar{\psi}_R\psi_R)\,\frac{\partial^4\mathcal{K}}{\partial S^2\partial\bar{S}^2}\Big]\,.
\end{aligned}
\tag{3.21}
$$

Now, to write the action in the standard form, we perform a partial integration on the $\frac{\partial\mathcal{K}}{\partial\bar{S}}$ term. The action so obtained is given below:

$$
\begin{aligned}
I_0' = \int d^2x \Big[\Big(& -\frac{1}{2}\partial_{\xi^-}A\partial_{\zeta^-}\bar{A} - \frac{1}{2}\partial_{\zeta^-}A\partial_{\xi^-}\bar{A} + i\bar{\psi}_L\partial_{\zeta^-}\psi_L + i\bar{\psi}_R\partial_{\xi^-}\psi_R - F\bar{F}\Big)\frac{\partial^2\mathcal{K}}{\partial S\partial\bar{S}} + \\
& + i\,(\bar{\psi}_L\bar{\psi}_R\bar{F})\,\frac{\partial^3\mathcal{K}}{\partial S^2\partial\bar{S}} + i\,(\,\psi_L\psi_R F + \bar{\psi}_L\psi_L\partial_{\zeta^-}\bar{A} + \bar{\psi}_R\psi_R\partial_{\xi^-}\bar{A})\,\frac{\partial^3\mathcal{K}}{\partial S\partial\bar{S}^2} - \\
& - (\bar{\psi}_L\psi_L\bar{\psi}_R\psi_R)\,\frac{\partial^4\mathcal{K}}{\partial S^2\partial\bar{S}^2}\Big]\,.
\end{aligned}
\tag{3.22}
$$

This should be compared with the usual $C=0$, $\mathcal{N}=2$ supersymmetric action [62], which can be determined by taking the superspace coordinates to anticommutate.

Before proceeding to write down the action, we first define the chiral and antichiral superfields in the $C=0$ *anticommutative* theory:

$$
\begin{aligned}
\tilde{S}(\xi^-,\zeta^-,\theta,\chi) &= \tilde{A}(\xi,\zeta) + i\theta\bar{\tilde{\psi}}_L(\xi,\zeta) - i\chi\bar{\tilde{\psi}}_R(\xi,\zeta) - \frac{i}{2}\theta\bar{\theta}\partial_\xi\tilde{A} - \frac{i}{2}\chi\bar{\chi}\partial_\zeta\tilde{A} + \\
&\quad + \theta\chi\left(\frac{\bar{\chi}}{2}\partial_\zeta\bar{\tilde{\psi}}_L + \frac{\bar{\theta}}{2}\partial_\xi\bar{\tilde{\psi}}_R - i\tilde{F} + \frac{1}{4}\bar{\theta}\bar{\chi}\partial_\xi\partial_\zeta\tilde{A}\right)
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
\bar{\tilde{S}}(\xi^+,\zeta^+,\bar{\theta},\bar{\chi}) &= \bar{\tilde{A}}(\xi,\zeta) - i\bar{\theta}\tilde{\psi}_L + i\bar{\chi}\tilde{\psi}_R + \frac{i}{2}\theta\bar{\theta}\partial_\xi\bar{\tilde{A}} + \frac{i}{2}\chi\bar{\chi}\partial_\zeta\bar{\tilde{A}} + \\
&\quad + \bar{\theta}\bar{\chi}\left(-\frac{\chi}{2}\partial_\zeta\tilde{\psi}_L - \frac{\theta}{2}\partial_\xi\tilde{\psi}_R - i\bar{\tilde{F}} + \frac{1}{4}\theta\chi\partial_\xi\partial_\zeta\bar{\tilde{A}}\right).
\end{aligned}
\tag{3.24}
$$

Hence, using the definitions of chiral and antichiral superfields given in eqns. (3.23) and (3.24), the classical action can be constructed by expanding the Kähler potential of the *anticommutative* theory. Below, we only give the final result for the action, after performing partial integrations on the terms of the type $\frac{\partial\mathcal{K}}{\partial\tilde{S}}$ and $\frac{\partial\mathcal{K}}{\partial\bar{\tilde{S}}}$:

$$
\begin{aligned}
\tilde{I}_0 = \int d^2x \Big[\Big\{ & -\frac{1}{2}\partial_\xi\tilde{A}\partial_\zeta\bar{\tilde{A}} - \frac{1}{2}\partial_\zeta\tilde{A}\partial_\xi\bar{\tilde{A}} + \frac{i}{2}\Big(\bar{\tilde{\psi}}_L\partial_\zeta\tilde{\psi}_L - \partial_\zeta\bar{\tilde{\psi}}_L\tilde{\psi}_L\Big) + \\
& + \frac{i}{2}\Big(\bar{\tilde{\psi}}_R\partial_\xi\tilde{\psi}_R - \partial_\xi\bar{\tilde{\psi}}_R\tilde{\psi}_R\Big) - F\bar{F}\ \Big\}\frac{\partial^2\mathcal{K}}{\partial\tilde{S}\partial\bar{\tilde{S}}} + \\
& + i\left(\ \bar{\tilde{\psi}}_L\bar{\tilde{\psi}}_R\bar{F} - \frac{1}{2}\bar{\tilde{\psi}}_L\tilde{\psi}_L\partial_\zeta\tilde{A} - \frac{1}{2}\bar{\tilde{\psi}}_R\tilde{\psi}_R\partial_\xi\tilde{A}\ \right)\ \frac{\partial^3\mathcal{K}}{\partial\tilde{S}^2\partial\bar{\tilde{S}}} + \\
& + i\Big(\ \tilde{\psi}_L\tilde{\psi}_R F + \frac{1}{2}\bar{\tilde{\psi}}_L\tilde{\psi}_L\partial_\zeta\bar{\tilde{A}} + \frac{1}{2}\bar{\tilde{\psi}}_R\tilde{\psi}_R\partial_\xi\bar{\tilde{A}}\ \Big)\ \frac{\partial^3\mathcal{K}}{\partial\tilde{S}\partial\bar{\tilde{S}}^2} - \\
& - \Big(\ \bar{\tilde{\psi}}_L\tilde{\psi}_L\bar{\tilde{\psi}}_R\tilde{\psi}_R\ \Big)\ \frac{\partial^4\mathcal{K}}{\partial\tilde{S}^2\partial\bar{\tilde{S}}^2}\Big]\,.
\end{aligned}
\tag{3.25}
$$

One can notice a slight mismatch in the form of I_0' given in (3.22) and $\tilde{I}_0$ given in (3.25). In particular, the A,$\bar{A}$ and $\tilde{A}$,$\bar{\tilde{A}}$ dependent terms with coefficient $\frac{\partial^3\mathcal{K}}{\partial\tilde{S}^2\partial\bar{\tilde{S}}}$ and $\frac{\partial^3\mathcal{K}}{\partial\tilde{S}\partial\bar{\tilde{S}}^2}$ in eqn. (3.25), are different in both the cases. However, the equivalence of the actions I_0' and $\tilde{I}_0$, up to a total derivative, can be explicitly seen by adding the following total derivative terms to the action in eqn. (3.25):

$$\frac{i}{2}\partial_\zeta\left(\bar{\tilde{\psi}}_L\tilde{\psi}_L\frac{\partial^2\mathcal{K}}{\partial\tilde{S}\partial\bar{\tilde{S}}}\right)+\frac{i}{2}\partial_\xi\left(\bar{\tilde{\psi}}_R\tilde{\psi}_R\frac{\partial^2\mathcal{K}}{\partial\tilde{S}\partial\bar{\tilde{S}}}\right). \tag{3.26}$$

The apparent difference in the two actions is only because I_0' is written in chiral coordinates (ξ^-,ζ^-), and $\tilde{I}_0$ is written in ordinary coordinates (ξ,ζ).

Coming to the non(anti)commutative part of the action I_C^n, one has to follow similar steps as discussed above. Using the results derived in eqns. (3.14)–(3.18), various terms in the expansion of the Kähler potential in the action in (3.1), which can contribute to terms in I_C^n are summarized below:

$$\begin{aligned} I_C^n = &\int d^2x d\theta d\chi d\bar{\theta} d\bar{\chi}\times \qquad\qquad (3.27)\\ &\times\sum_{n=1}^{\infty}\Bigg[\frac{1}{(2n+1)!}\left[L_*^{2n}*R\right]\frac{\partial^{2n+1}\mathcal{K}}{\partial S^{2n}\partial\bar{S}}+\frac{1}{(2n+2)!}\left[L_*^{2n+1}*R\right]\frac{\partial^{2n+2}\mathcal{K}}{\partial S^{2n+1}\partial\bar{S}}+\\ &\quad+\frac{1}{(2n+2)!}\left[L_*^{2n}*R_*^2\right]\frac{\partial^{2n+2}\mathcal{K}}{\partial S^{2n}\partial\bar{S}^2}+\frac{1}{(2n+3)!}\left[L_*^{2n+1}*R_*^2\right]\frac{\partial^{2n+3}\mathcal{K}}{\partial S^{2n+1}\partial\bar{S}^2}\Bigg].\end{aligned}$$

Now, it is straightforward to substitute the results derived in eqns.(3.14)–(3.18), and collect the terms proportional to various powers of $(\det C)$. Then, performing an integration over the grassmannian variables in the usual way, one will end up with the final form of action. In this context we first collect terms which are to lowest order in $(\det C)$ and write down the action explicitly to first order in $(\det C)$. The subsequent generalization to include terms proportional to arbitrary powers of $(\det C)$ will follow.

The non(anti)commutative part of the action obtained from eqn. (3.27) to first order in $(\det C)$ (labeled as I_{C1}) turns out to be:

$$\begin{aligned} I_{C1} = \left(\frac{1}{4}\det C\right)\int d^2x\Bigg[&\left(\frac{1}{2}F^2\partial_{\xi^-}\partial_{\zeta^-}\bar{A}\right)\frac{\partial^3\mathcal{K}}{\partial S^2\partial\bar{S}}+\frac{F}{6}\times\\ &\times\left(-2i\bar{\psi}_L\bar{\psi}_R\partial_{\xi^-}\partial_{\zeta^-}\bar{A}+i\bar{\psi}_L\partial_{\zeta^-}\psi_LF+i\bar{\psi}_R\partial_{\xi^-}\psi_RF-F^2\bar{F}\right)\times\\ &\times\frac{\partial^4\mathcal{K}}{\partial S^3\partial\bar{S}}+\frac{1}{2}\left(F^2\partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A}\right)\frac{\partial^4\mathcal{K}}{\partial S^2\partial\bar{S}^2}+\\ &+\frac{i}{6}\left(\bar{\psi}_L\bar{\psi}_RF^2\bar{F}\right)\frac{\partial^5\mathcal{K}}{\partial S^4\partial\bar{S}}-\frac{i}{6}F\times\\ &\times\left(2\bar{\psi}_L\bar{\psi}_R\partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A}-F^2\psi_L\psi_R+F\psi_L\bar{\psi}_L\partial_{\zeta^-}\bar{A}+F\psi_R\bar{\psi}_R\partial_{\xi^-}\bar{A}\right)\times\\ &\times\frac{\partial^5\mathcal{K}}{\partial S^3\partial\bar{S}^2}-\frac{1}{6}\left(F^2\bar{\psi}_L\psi_L\bar{\psi}_R\psi_R\right)\frac{\partial^6\mathcal{K}}{\partial S^4\partial\bar{S}^2}\Bigg].\end{aligned} \tag{3.28}$$

As was first pointed in [13], C by itself breaks Lorentz invariance, but since it appears in the action (3.28), as $(\det C)$ no Lorentz invariance is broken. Hence, the non(anti)commutative part of the action (3.28), is Lorentz invariant to first order in $(\det C)$.

Having written the action to first order in $(\det C)$ in eqn. (3.28), we now proceed to write the most general action, by following similar steps as discussed above. We substitute the results obtained in eqns. (3.14)-(3.18), in the action (3.1), and collect terms which are proportional to $(\det C)^n$. The most general action for non(anti)commutative part of the $\mathcal{N}=2$ supersymmetric theory, then looks as follows:

$$\begin{aligned}
I_C^n &= \sum_{n=2}^{\infty}\left(\frac{1}{4}\det C\right)^{n-1}\int d^2x F^{2n-2}\frac{2n}{(2n)!}\left[i\bar{\psi}_L\bar{\psi}_R\bar{F}\frac{\partial^{2n+1}\mathcal{K}}{\partial S^{2n}\partial\bar{S}} - \bar{\psi}_L\psi_L\bar{\psi}_R\psi_R\frac{\partial^{2n+2}\mathcal{K}}{\partial S^{2n}\partial\bar{S}^2}\right] + \\
&\quad + \sum_{n=1}^{\infty}\left(\frac{1}{4}\det C\right)^{n}\int d^2x F^{2n-2}\times \\
&\quad \times\left[\frac{1}{(2n)!}F^2\partial_{\xi^-}\partial_{\zeta^-}\bar{A}\frac{\partial^{2n+1}\mathcal{K}}{\partial S^{2n}\partial\bar{S}} - \frac{F}{(2n+1)!}\Big(2ni\bar{\psi}_L\bar{\psi}_R\partial_{\xi^-}\partial_{\zeta^-}\bar{A} - i\bar{\psi}_L\partial_{\zeta^-}\psi_L F - \right. \\
&\qquad\qquad - i\bar{\psi}_R\partial_{\xi^-}\psi_R F + F^2\bar{F}\Big)\frac{\partial^{2n+2}\mathcal{K}}{\partial S^{2n+1}\partial\bar{S}} + \\
&\quad + \frac{1}{(2n)!}F^2\partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A}\frac{\partial^{2n+2}\mathcal{K}}{\partial S^{2n}\partial\bar{S}^2} - \frac{i}{(2n+1)!}\times \\
&\quad \left.\times F\Big(2n\bar{\psi}_L\bar{\psi}_R\partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A} - F^2\psi_L\psi_R + F\psi_L\bar{\psi}_L\partial_{\zeta^-}\bar{A} + F\psi_R\bar{\psi}_R\partial_{\xi^-}\bar{A}\Big)\frac{\partial^{2n+3}\mathcal{K}}{\partial S^{2n+1}\partial\bar{S}^2}\right].
\end{aligned} \tag{3.29}$$

One can clearly see a power series expansion in $(\det C)$. The full action for the $\mathcal{N}=2$ supersymmetric theory on a non(anti)commutative superspace is thus given by eqns. (3.22) and (3.29). We now proceed to analyze the supersymmetry of the theory.

4. Supersymmetry

The two supersymmetries preserved by the usual $C=0, \mathcal{N}=2$ supersymmetric theories are generated by the supercharges Q and $\bar{Q}$. However, from the discussion in section 1, we know that, once we invoke non(anti)commutativity of the superspace coordinates as in eqn. (2.5), the $\bar{Q}$ symmetry is broken and the superspace becomes $N=\frac{2}{2}$ supersymmetric. Accordingly, we shall use the unbroken Q symmetry on the superfields to generate the supersymmetry transformations.

Thus, the variation of the chiral superfield in eqn. (2.28) is given as:

$$\delta_\alpha S = \delta_\alpha A + i\theta\delta_\alpha\bar{\psi}_L - i\chi\delta_\alpha\bar{\psi}_R - i(\theta\chi)\delta_\alpha F\,. \tag{4.1}$$

As stated before, we have suppressed the explicit dependence of component fields on the coordinates for clarity. All the component fields are taken to be functions of (ξ^-,ζ^-) unless specified otherwise. The supercharge Q acts on the chiral superfield as shown:

$$\begin{aligned}
\delta_\alpha S &= (\alpha_1 Q_1 + \alpha_2 Q_2\ ) * S\,, \\
&= -\alpha_1\left(i\bar{\psi}_L - i\chi F\right) + \alpha_2\left(-i\bar{\psi}_R + i\theta F\right).
\end{aligned} \tag{4.2}$$

Now, it is easy to compare eqns. (4.1) and (4.2), and get the supersymmetry transformations for the component fields:

$$\begin{aligned}
\delta_\alpha A &= -i\alpha_1\bar{\psi}_L - i\alpha_2\bar{\psi}_R\,, \\
\delta_\alpha \bar{\psi}_L &= -\alpha_2 F\,, \\
\delta_\alpha \bar{\psi}_R &= \alpha_1 F\,, \\
\delta_\alpha F &= 0\,.
\end{aligned} \tag{4.3}$$

Analogously, the supersymmetry transformations of the remaining component fields appearing in the antichiral superfield in eqn. (2.31) can also be written:

$$\begin{aligned}
\delta_\alpha \bar{A} &= 0\,, \\
\delta_\alpha \psi_L &= -\alpha_1\partial_{\xi^-}\bar{A}\,, \\
\delta_\alpha \psi_R &= -\alpha_2\partial_{\zeta^-}\bar{A}\,, \\
\delta_\alpha \bar{F} &= i\alpha_1\partial_{\xi^-}\psi_R - i\alpha_2\partial_{\zeta^-}\psi_L\,.
\end{aligned} \tag{4.4}$$

Notice that the supersymmetry transformations given in eqns. (4.3) and (4.4) are the same as in the usual $C = 0$ theory. Also, we have already pointed out the equivalence of the $C = 0$ part of our action in eqn. (3.21) and the standard action [62] given in eqn. (3.25). Hence, the supersymmetry of the $C = 0$ part of the action (3.21), is obvious. However, one can also explicitly check the supersymmetry by varying the action (3.21), with respect to the supersymmetry parameters α_1 and α_2 and using the supersymmetry transformations (4.3) and (4.4). Thus, the variation of the action in eqn. (3.21) gives:

$$\delta I_0 = \int d^2x\,\left[I_0^1 + I_0^2 + I_0^3 + I_0^4\right], \tag{4.5}$$

where $I_0^1, I_0^2, I_0^3, I_0^4$ are defined below as:

$$\begin{aligned}
I_0^1 &= \Big[\partial_{\xi^-}\partial_{\zeta^-}\bar{A}(\delta A) - i\partial_{\zeta^-}(\delta\psi_L)\bar{\psi}_L - i\partial_{\zeta^-}\psi_L(\delta\bar{\psi}_L) - \\
&\qquad - i\partial_{\xi^-}(\delta\psi_R)\bar{\psi}_R - i\partial_{\xi^-}\psi_R(\delta\bar{\psi}_R) - F(\delta\bar{F})\Big]\frac{\partial^2\mathcal{K}}{\partial S\partial\bar{S}}\,, \qquad (4.6)\\
I_0^2 &= -\Big[\Big(i\partial_{\zeta^-}\psi_L\bar{\psi}_L + i\partial_{\xi^-}\psi_R\bar{\psi}_R + F\bar{F}\Big)(\delta A) + \\
&\qquad + i(\delta\bar{\psi}_R)\bar{\psi}_L\bar{F} + i\bar{\psi}_R(\delta\bar{\psi}_L)\bar{F} + i\bar{\psi}_R\bar{\psi}_L(\delta\bar{F})\Big]\frac{\partial^3\mathcal{K}}{\partial S^2\partial\bar{S}}\,, \qquad (4.7)\\
I_0^3 &= \Big[i(\delta\psi_L)\psi_R F + i\psi_L(\delta\psi_R)F + i(\delta\bar{\psi}_L)\psi_L\partial_{\zeta^-}\bar{A} + \Big(\partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A}\Big)(\delta A) + \\
&\qquad + i\bar{\psi}_L(\delta\psi_L)\partial_{\zeta^-}\bar{A} + i(\delta\bar{\psi}_R)\psi_R\partial_{\xi^-}\bar{A} + i\bar{\psi}_R(\delta\psi_R)\partial_{\xi^-}\bar{A}\Big]\frac{\partial^3\mathcal{K}}{\partial S\partial\bar{S}^2}\,, \qquad (4.8)\\
I_0^4 &= -\Big[\Big((\delta\bar{\psi}_L)\psi_L\bar{\psi}_R\psi_R + \bar{\psi}_L(\delta\psi_L)\bar{\psi}_R\psi_R + \bar{\psi}_L\psi_L(\delta\bar{\psi}_R)\psi_R + \bar{\psi}_L\psi_L\bar{\psi}_R(\delta\psi_R)\Big) - \\
&\qquad - i\Big(\psi_L\psi_R F + \bar{\psi}_L\psi_L\partial_{\zeta^-}\bar{A} + \bar{\psi}_R\psi_R\partial_{\xi^-}\bar{A}\Big)(\delta A)\Big]\frac{\partial^4\mathcal{K}}{\partial S^2\partial\bar{S}^2}\,. \qquad (4.9)
\end{aligned}$$

Now, one can directly substitute the supersymmetry transformations given in eqns. (4.3) and (4.4), in the variation of the action given in eqn. (4.5), or equivalently in eqns. (4.6)–(4.9). The result is that, each of the terms in eqns. (4.6)–(4.9) vanishes individually.

Therefore, the $C=0$ part of the action given in eqn. (3.21) preserves $N=2/2$ supersymmetry. Also, by writing down transformations under both Q and $\bar{Q}$'s, we find that the supersymmetry variation corresponding to Q's in the action I_0 cancels out, whereas the C dependent terms generated by $\bar{Q}'$s add up. Therefore, the action does not remain invariant under $\bar{Q}$'s.

Now, regarding the non(anti)commutative part of the action in eqn. (3.29), we will first check the supersymmetry of the action (3.28) to first order in $(\det C)$. The calculations get simplified by noting that the supersymmetry transformations of the component fields F and $\bar{A}$ vanish identically. Thus, varying the action (3.28), with respect to the parameters α_1 and α_2, and leaving out the terms which cancel out trivially, we end up with:

$$\delta I_{C1} = \left(\frac{1}{4}\det C\right)\int d^2x\,\left[I_C^1 + I_C^2 + I_C^3 + I_C^4\right], \tag{4.10}$$

where $I_C^1, I_C^2, I_C^3, I_C^4$ are defined below as:

$$\begin{aligned} I_C^1 = &\left[\frac{1}{2}F^2\partial_{\xi^-}\partial_{\zeta^-}\bar{A}(\delta A) + \frac{i}{6}\Big(-2F(\delta\bar{\psi}_L)\bar{\psi}_R\partial_{\xi^-}\partial_{\zeta^-}\bar{A} - 2F\bar{\psi}_L(\delta\bar{\psi}_R)\partial_{\xi^-}\partial_{\zeta^-}\bar{A} + \right.\\ &\qquad + F^2(\delta\bar{\psi}_L)\partial_{\zeta^-}\psi_L + F^2\bar{\psi}_L\partial_{\zeta^-}(\delta\psi_L) + F^2(\delta\bar{\psi}_R)\partial_{\xi^-}\psi_R + \\ &\qquad \left. + F^2\bar{\psi}_R\partial_{\xi^-}(\delta\psi_R) + iF^3\delta\bar{F}\Big)\right]\frac{\partial^4\mathcal{K}}{\partial S^3\partial\bar{S}}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} I_C^2 = &\left[\frac{1}{6}F\Big\{-2i\bar{\psi}_L\bar{\psi}_R\partial_{\xi^-}\partial_{\zeta^-}\bar{A} + i\bar{\psi}_L\partial_{\zeta^-}\psi_L F + i\bar{\psi}_R\partial_{\xi^-}\psi_R F - F^2\bar{F}\Big\}(\delta A) + \right.\\ &\left. + \frac{i}{6}F^2\left\{(\delta\bar{\psi}_L)\bar{\psi}_R\bar{F} + \bar{\psi}_L(\delta\bar{\psi}_R)\bar{F} + \bar{\psi}_L\bar{\psi}_R(\delta\bar{F})\right\}\right]\frac{\partial^5\mathcal{K}}{\partial S^4\partial\bar{S}}, \end{aligned} \tag{4.12}$$

$$\begin{aligned} I_C^3 = &\left[\frac{1}{2}F^2\partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A}(\delta A) - \frac{i}{6}F\Big\{2(\delta\bar{\psi}_L)\bar{\psi}_R\partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A} + 2\bar{\psi}_L(\delta\bar{\psi}_R)\partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A} - \right.\\ &\qquad - F^2(\delta\psi_L)\psi_R - F^2\psi_L(\delta\psi_R) + F(\delta\psi_L)\bar{\psi}_L\partial_{\zeta^-}\bar{A} + \\ &\qquad + F\psi_L(\delta\bar{\psi}_L)\partial_{\zeta^-}\bar{A} + F(\delta\psi_R)\bar{\psi}_R\partial_{\xi^-}\bar{A} + \\ &\qquad \left. + F\psi_R(\delta\bar{\psi}_R)\partial_{\xi^-}\bar{A}\Big\}\right]\frac{\partial^5\mathcal{K}}{\partial S^3\partial\bar{S}^2} \end{aligned} \tag{4.13}$$

$$\begin{aligned} I_C^4 = -&\left[\frac{i}{6}F\Big(2\bar{\psi}_L\bar{\psi}_R\partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A} - F^2\psi_L\psi_R - F\psi_L\bar{\psi}_L\partial_{\zeta^-}\bar{A} - F\psi_R\bar{\psi}_R\partial_{\xi^-}\bar{A}\Big)(\delta A)\right.\\ &+ \frac{1}{6}\Big(F^2(\delta\bar{\psi}_L)\psi_L\bar{\psi}_R\psi_R + F^2\bar{\psi}_L(\delta\psi_L)\bar{\psi}_R\psi_R + F^2\bar{\psi}_L\psi_L(\delta\bar{\psi}_R)\psi_R + \\ &\qquad \left. + F^2\bar{\psi}_L\psi_L\bar{\psi}_R(\delta\psi_R)\Big)\right]\frac{\partial^6\mathcal{K}}{\partial S^4\partial\bar{S}^2}. \end{aligned} \tag{4.14}$$

Now, one can directly substitute the supersymmetry transformations of the component fields given in (4.3) and (4.4), in the variation of the action given in eqn. (4.10) or equivalently in eqns. (4.11)–(4.14). The outcome being that, each of the terms in the action given in eqns. (4.11)–(4.14), vanish individually. Thus, we have the result that the non(anti)commutative part of the action given in eqn. (3.28), preserves $N=2/2$ supersymmetry to first order in $(\det C)$.

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To check whether the non(anti)commutative part of the full action given in eqn. (3.29) preserves $N = \frac{2}{2}$ supersymmetry to all orders in $(\det C)$, we perform an explicit verification of supersymmetry. We follow the procedure as discussed above and vary the action I_C given in eqn. (3.29) with respect to the supersymmetry parameters α_1 and α_2. It turns out that the supersymmetry variation of I_C^n again consists of four parts (ignoring terms cancelling trivially):

$$\delta I_C^n = \left(\frac{1}{4}\det C\right)^n \int d^2x F^{2n-2}\left[I^1_{Cn} + I^2_{Cn} + I^3_{Cn} + I^4_{Cn}\right], \tag{4.15}$$

where $I^1_{Cn}, I^2_{Cn}, I^3_{Cn}, I^4_{Cn}$ are defined below as:

$$\begin{aligned}
I^1_{Cn} &= \Big[\frac{1}{(2n)!}F^2\partial_{\xi^-}\partial_{\zeta^-}\bar{A}(\delta A) + \frac{i}{(2n+1)!}\times \\
&\quad\times\Big(-2nF(\delta\bar\psi_L)\bar\psi_R\partial_{\xi^-}\partial_{\zeta^-}\bar A - 2nF\bar\psi_L(\delta\bar\psi_R)\partial_{\xi^-}\partial_{\zeta^-}\bar A + F^2(\delta\bar\psi_L)\partial_{\zeta^-}\psi_L + \\
&\qquad + F^2\bar\psi_L\partial_{\zeta^-}(\delta\psi_L) + F^2(\delta\bar\psi_R)\partial_{\xi^-}\psi_R + F^2\bar\psi_R\partial_{\xi^-}(\delta\psi_R) + iF^3\delta\bar F\Big)\Big]\frac{\partial^{2n+2}\mathcal{K}}{\partial S^{2n+1}\partial\bar S},
\end{aligned}\tag{4.16}$$

$$\begin{aligned}
I^2_{Cn} &= \frac{F}{(2n+1)!}\Big[\Big(-2in\bar\psi_L\bar\psi_R\partial_{\xi^-}\partial_{\zeta^-}\bar A + i\bar\psi_L\partial_{\zeta^-}\psi_L F + i\bar\psi_R\partial_{\xi^-}\psi_R F - F^2\bar F\Big)(\delta A) + \\
&\qquad + i(\delta\bar\psi_L)\bar\psi_R F\bar F + i\bar\psi_L(\delta\bar\psi_R)F\bar F + i\bar\psi_L\bar\psi_R F(\delta\bar F)\Big]\frac{\partial^{2n+3}\mathcal{K}}{\partial S^{2n+2}\partial\bar S},
\end{aligned}\tag{4.17}$$

$$\begin{aligned}
I^3_{Cn} &= \Big[\frac{1}{(2n)!}F^2\partial_{\xi^-}\bar A\partial_{\zeta^-}\bar A(\delta A) - \frac{i}{(2n+1)!}F\times \\
&\quad\times\Big\{2n(\delta\bar\psi_L)\bar\psi_R\partial_{\xi^-}\bar A\partial_{\zeta^-}\bar A + 2n\bar\psi_L(\delta\bar\psi_R)\partial_{\xi^-}\bar A\partial_{\zeta^-}\bar A - F^2(\delta\psi_L)\psi_R - F^2\psi_L(\delta\psi_R) + \\
&\qquad + F(\delta\psi_L)\bar\psi_L\partial_{\zeta^-}\bar A + F\psi_L(\delta\bar\psi_L)\partial_{\zeta^-}\bar A + F(\delta\psi_R)\bar\psi_R\partial_{\xi^-}\bar A + F\psi_R(\delta\bar\psi_R)\partial_{\xi^-}\bar A\Big\}\Big]\times \\
&\quad\times\frac{\partial^{2n+3}\mathcal{K}}{\partial S^{2n+1}\partial\bar S^2}
\end{aligned}\tag{4.18}$$

$$\begin{aligned}
I^4_{Cn} &= -\frac{1}{(2n+1)!}\Big[iF\Big(2n\bar\psi_L\bar\psi_R\partial_{\xi^-}\bar A\partial_{\zeta^-}\bar A - F^2\psi_L\psi_R - F\psi_L\bar\psi_L\partial_{\zeta^-}\bar A - F\psi_R\bar\psi_R\partial_{\xi^-}\bar A\Big)(\delta A) + \\
&\qquad + \Big(F^2(\delta\bar\psi_L)\psi_L\bar\psi_R\psi_R + F^2\bar\psi_L(\delta\psi_L)\bar\psi_R\psi_R + F^2\bar\psi_L\psi_L(\delta\bar\psi_R)\psi_R + \\
&\qquad + F^2\bar\psi_L\psi_L\bar\psi_R(\delta\psi_R)\Big)\Big]\frac{\partial^{2n+4}\mathcal{K}}{\partial S^{2n+2}\partial\bar S^2}
\end{aligned}\tag{4.19}$$

Now, one can go on to substitute the supersymmetry variation of the component fields given in eqns. (4.3) and (4.4) and hence in eqns. (4.16)–(4.19). Owing to some beautiful cancellations, all the pieces of the supersymmetry variation of the full action I_C given in eqn. (4.15) vanish identically. Hence, the non(anti)commutative part of the supersymmetric theory given in eqn. (3.29) preserves $N = 2/2$ supersymmetry to all orders in $(\det C)$.

Combining the results obtained for the supersymmetry variations of the actions given in eqns. (3.21) and (3.29), we conclude that the full non(anti)commutative theory given by eqn. (3.19), preserves $N = \frac{2}{2}$ supersymmetry.

4.1 Superpotential

In this section, we generalize the results of the preceding section, by including superpotentials in the action (3.1). The complete kinetic action for the non(anti)commutative $N = \frac{2}{2}$ theory was given in eqn. (3.19). The action can be generalized to include arbitrary superpotentials, as shown :

$$I_g = \int d^2x d\theta d\chi d\bar{\theta} d\bar{\chi} \mathcal{K}(S, \bar{S}) + \int d^2x d\theta d\chi W(S) + \int d^2x d\bar{\theta} d\bar{\chi} \bar{W}(\bar{S}) , \tag{4.20}$$

where apart from the Kähler potential $\mathcal{K}(S, \bar{S})$, we also have superpotentials $W(S)$ and $\bar{W}(\bar{S})$. Like the case for $\mathcal{K}(S, \bar{S})$ in eqn. (3.2), we expand the superpotentials in terms of the component fields as shown:

$$\begin{aligned} W(S) &= W(A) + L \frac{\partial W}{\partial S}|_{S=A} + \frac{1}{2!} L * L \frac{\partial^2 W}{\partial S^2}|_{S=A} + \frac{1}{3!} L * L * L \frac{\partial^3 W}{\partial S^3}|_{S=A} + \\ &+ \cdots + \frac{1}{(2n)!} L_*^{2n} \frac{\partial^{2n} W}{\partial S^{2n}}|_{S=A} + \cdots + \frac{1}{(2n+1)!} L_*^{2n+1} \frac{\partial^{2n+1} W}{\partial S^{2n+1}}|_{S=A} + \cdots . \end{aligned} \tag{4.21}$$

Note that the quantities L, L_*^{2n} and L_*^{2n+1} have been defined in eqns. (3.3), (3.8) and (3.9) respectively. We also have:

$$\bar{W}(\bar{S}) = \bar{W}(\bar{A}) + R \frac{\partial \bar{W}}{\partial \bar{S}}|_{\bar{S}=\bar{A}} + \frac{1}{2!} R * R \frac{\partial^2 \bar{W}}{\partial \bar{S}^2}|_{\bar{S}=\bar{A}} , \tag{4.22}$$

where R appearing above, has been defined in eqn. (3.3). Notice that the above expansion of $\bar{W}(\bar{S})$ gets truncated, since $R_*^n = 0$, for $n \geq 3$. Hence, using the results of section 3, we can evaluate various terms in the superpotential in eqn. (4.21). Collecting terms proportional to $\theta\chi$ which will ultimately contribute to the action, we end up with:

$$\begin{aligned} W(S)|_{\theta\chi} &= -iF \frac{\partial W}{\partial S}|_{S=A} - \bar{\psi}_L \bar{\psi}_R \frac{\partial^2 W}{\partial S^2}|_{S=A} - \\ &- \sum_{n=2}^{\infty} \frac{1}{(2n-1)!} \left(\frac{1}{4} \det C \right)^{n-1} F^{2n-2} \bar{\psi}_L \bar{\psi}_R \frac{\partial^{2n} W}{\partial S^{2n}}|_{S=A} - \\ &- \sum_{n=1}^{\infty} \frac{i}{(2n+1)!} \left(\frac{1}{4} \det C \right)^{n} F^{2n+1} \frac{\partial^{2n+1} W}{\partial S^{2n+1}}|_{S=A} + \cdots . \end{aligned} \tag{4.23}$$

Similarly, one can also write down the terms coming from eqn. (4.22). We end up with the following most general form of $\bar{W}(\bar{S})$, after collecting the terms proportional to $\bar{\theta}\bar{\chi}$:

$$\bar{W}(\bar{S})|_{\bar{\theta}\bar{\chi}} = -i\bar{F} \frac{\partial \bar{W}}{\partial \bar{S}}|_{\bar{S}=\bar{A}} - \psi_L \psi_R \frac{\partial^2 \bar{W}}{\partial \bar{S}^2}|_{\bar{S}=\bar{A}} . \tag{4.24}$$

5. Discussion

To conclude, we have analyzed the most general classical action of an $\mathcal{N} = 2$ supersymmetric theory with single chiral and antichiral superfield, defined on a non(anti)commutative superspace, for an arbitrary Kähler and super potential. The key aspect of the analysis is the emergence of a power series expansion in $(\det C)$ in the action. Writing down a general n^{th} order term in the action is possible and is found to be Lorentz invariant. The $N = 2/2$ supersymmetry of the action was explicitly shown, order by order in $(\det C)$.

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The $C = 0$ part of the supersymmetric theory studied in this paper, has a straightforward generalization to supersymmetric nonlinear σ-model containing several superfields. From the definitions of geometric quantities given in section 2.2, the action I_0 given in eqn. (3.22), can be written as [62]:

$$
\begin{aligned}
I_0 = \Bigg[\int d^2x \; g_{i\bar{j}} \Big(& - \partial_{\xi^-} A^i \partial_{\zeta^-} \bar{A}^j + i\bar{\psi}^i_L \partial_{\zeta^-} \psi^j_L + i\bar{\psi}^i_R \partial_{\xi^-} \psi^j_R - F^i \bar{F}^j \Big) + \\
& + \Gamma_{ij\bar{k}} \Big(\bar{\psi}^i_L \bar{\psi}^j_R \bar{F}^k \Big) + i\Gamma_{\bar{i}\bar{j}k} \Big(\psi^i_L \psi^j_R F^k - \bar{\psi}^k_L \psi^i_L \partial_{\zeta^-} \bar{A}^j - \bar{\psi}^k_R \psi^i_R \partial_{\xi^-} \bar{A}^j \Big) - \\
& - R_{\bar{i}j\bar{k}l} \Big(\psi^i_L \bar{\psi}^j_L \psi^k_R \bar{\psi}^l_R \Big) \Bigg] .
\end{aligned} \tag{5.1}
$$

It seems plausible that the action for the complete $N = \frac{2}{2}$ supersymmetric theory with arbitrary number of superfields S^i and $\bar{S}^i$ can also be written in terms of the Kähler metric and the subsequent geometric quantities following from it. However, to geometrize the full $N = \frac{2}{2}$ supersymmetric theory studied in this paper, the identities given eqns. (2.39) and (2.40), are not enough. Also terms of the type $\psi^i_L \cdots \psi^n_R$ etc., will appear in the expressions such as L^n_*. It will be interesting to examine whether the complete action can be written in a geometric form like (5.1).

There are lots of other avenues one can explore. One of the most obvious things to study is the quantum aspects of the non(anti)commutative theory we have discussed. It is a well acclaimed fact that the Kähler geometry imposes severe restrictions on the kind of counter terms which can appear in the quantum action. In view of the series expansion we see in the classical action, it is worthwhile to analyze what kind of restrictions put by Kähler geometry appear at the quantum level. Moreover, checking the renormalizability of the above model is an interesting aspect to be pursued.

Further, in two dimensions one can have new σ-models due to the possibility of defining twisted multiplets [60], in addition to the chiral and antichiral multiplets. These σ-models have established various connections between $D = 4$, $\mathcal{N} = 1$ and $D = 2$, $\mathcal{N} = 2$ models. It may be possible to generalize the result of this paper, to include twisted (chiral and antichiral) fields as well. In this way, one can possibly study the consequences of Mirror symmetry in $D = 2$, $N = (2, 2)$ models, which interchanges the (anti)chiral and twisted (anti)chiral fields.

A. Proof of relevant identities

Here, we provide the proof of the identity:

$$
L^m_* * R * L^n_* * R * L^p_*|_{\bar{\theta}\bar{\chi}\theta\chi} = L^{(m+n+p)}_* * R^2_*|_{\bar{\theta}\bar{\chi}\theta\chi} \,, \tag{A.1}
$$

where m, n and p are integers. As already mentioned, the above identity strictly holds only for terms with coefficient $\bar{\theta}\bar{\chi}(\theta\chi)$. Before proceeding, below we collect certain results from the text, which will be used repeatedly:

$$
L = +i\theta\bar{\psi}_L - i\chi\bar{\psi}_R - i(\theta\chi)F \,, \tag{A.2}
$$

$$
\begin{aligned}
R &= -i\bar\theta\psi_L + i\bar\chi\psi_R + i\theta\bar\theta\partial_{\xi^-}\bar A + i\chi\bar\chi\partial_{\zeta^-}\bar A + \\
&\quad + \bar\theta\bar\chi\left(-\chi\partial_{\zeta^-}\psi_L - \theta\partial_{\xi^-}\psi_R - i\bar F + (\theta\chi)\partial_{\xi^-}\partial_{\zeta^-}\bar A\right), \qquad (A.3)\\
L_*^{2n} &= \left(\frac{1}{4}\det C\right)^{n-1} F^{2n-2}\left[-2n(\theta\chi)\bar\psi_L\bar\psi_R + \left(\frac{1}{4}\det C\right)F^2\right], \qquad (A.4)\\
L_*^{2n+1} &= \left(\frac{1}{4}\det C\right)^{n} F^{2n-1}\left(-2ni\bar\psi_L\bar\psi_R + F.L\right), \qquad (A.5)\\
R_*^2 &= 2\bar\theta\bar\chi\Big(-\psi_L\psi_R - \chi\psi_L\partial_{\zeta^-}\bar A - \theta\psi_R\partial_{\xi^-}\bar A + (\theta\chi)\partial_{\xi^-}\bar A\partial_{\zeta^-}\bar A\Big), \qquad (A.6)\\
R_*^n &= 0, \quad \text{for} \quad n>2. \qquad (A.7)
\end{aligned}
$$

To begin, using the definitions of L and R given in eqns. (A.2), (A.3) we deduce the following result:

$$
\begin{aligned}
R*L &= \frac{1}{2}\bar\theta\Big(C^{00}\bar\psi_L\partial_{\xi^-}\bar A + C^{01}\bar\psi_R\partial_{\xi^-}\bar A\Big) - \frac{1}{2}\bar\chi\Big(C^{01}\bar\psi_L\partial_{\zeta^-}\bar A + C^{11}\bar\psi_R\partial_{\zeta^-}\bar A\Big) - \\
&\quad - \bar\chi\theta\Big(\bar\psi_L\psi_R - \frac{1}{2}C^{11}F\partial_{\zeta^-}\bar A\Big) - \bar\theta\chi\Big(\bar\psi_R\psi_L + \frac{1}{2}C^{00}F\partial_{\xi^-}\bar A\Big) - \\
&\quad - \bar\theta\theta\Big(-\bar\psi_L\psi_L + \frac{1}{2}C^{01}F\partial_{\xi^-}\bar A\Big) + \bar\chi\chi\Big(\bar\psi_R\psi_R + \frac{1}{2}C^{10}F\partial_{\zeta^-}\bar A\Big) + \\
&\quad + \frac{i}{2}\bar\theta\bar\chi\Big(C^{10}\bar\psi_L\partial_{\zeta^-}\psi_L + C^{11}\bar\psi_R\partial_{\zeta^-}\psi_L - C^{00}\bar\psi_L\partial_{\xi^-}\psi_R - C^{01}\bar\psi_R\partial_{\xi^-}\psi_R + \\
&\qquad\qquad + \frac{1}{2}(\det C)F\partial_{\xi^-}\partial_{\zeta^-}\bar A\Big) - \bar\theta(\theta\chi)\left(\bar\psi_R\partial_{\xi^-}\bar A + F\psi_L\right) - \\
&\quad - \bar\chi(\theta\chi)\left(\bar\psi_L\partial_{\zeta^-}\bar A - F\psi_R\right) + \\
&\quad + \bar\theta\bar\chi\theta\Big\{\frac{i}{2}\Big(-C^{11}\partial_{\zeta^-}\psi_L F + C^{10}\partial_{\xi^-}\psi_R F - \\
&\qquad\qquad - C^{01}\bar\psi_L\partial_{\xi^-}\partial_{\zeta^-}\bar A - C^{11}\bar\psi_R\partial_{\xi^-}\partial_{\zeta^-}\bar A\Big) + \bar\psi_L\bar F\Big\} + \\
&\quad + \bar\theta\bar\chi\chi\Big\{\frac{i}{2}\Big(C^{00}\partial_{\xi^-}\psi_R F - C^{10}\partial_{\zeta^-}\psi_L F - \\
&\qquad\qquad - C^{01}\bar\psi_R\partial_{\xi^-}\partial_{\zeta^-}\bar A - C^{00}\bar\psi_L\partial_{\xi^-}\partial_{\zeta^-}\bar A\Big) - \bar\psi_R\bar F\Big\} + \\
&\quad + \bar\theta\bar\chi(\theta\chi)\left(i\bar\psi_L\partial_{\zeta^-}\psi_L + i\bar\psi_R\partial_{\xi^-}\psi_R - F\bar F\right). \qquad (A.8)
\end{aligned}
$$

One can similarly obtain $L*R$ as given below:

$$
\begin{aligned}
L*R &= -\frac{1}{2}\bar\theta\Big(C^{00}\bar\psi_L\partial_{\xi^-}\bar A + C^{01}\bar\psi_R\partial_{\xi^-}\bar A\Big) + \frac{1}{2}\bar\chi\Big(C^{01}\bar\psi_L\partial_{\zeta^-}\bar A + C^{11}\bar\psi_R\partial_{\zeta^-}\bar A\Big) - \\
&\quad - \bar\chi\theta\Big(\bar\psi_L\psi_R + \frac{1}{2}C^{11}F\partial_{\zeta^-}\bar A\Big) - \bar\theta\chi\Big(\bar\psi_R\psi_L - \frac{1}{2}C^{00}F\partial_{\xi^-}\bar A\Big) - \\
&\quad - \bar\theta\theta\Big(-\bar\psi_L\psi_L - \frac{1}{2}C^{01}F\partial_{\xi^-}\bar A\Big) + \bar\chi\chi\Big(\bar\psi_R\psi_R - \frac{1}{2}C^{10}F\partial_{\zeta^-}\bar A\Big) - \\
&\quad - \frac{i}{2}\bar\theta\bar\chi\Big(C^{10}\bar\psi_L\partial_{\zeta^-}\psi_L + C^{11}\bar\psi_R\partial_{\zeta^-}\psi_L - C^{00}\bar\psi_L\partial_{\xi^-}\psi_R - \\
&\qquad\qquad - C^{01}\bar\psi_R\partial_{\xi^-}\psi_R - \frac{1}{2}(\det C)F\partial_{\xi^-}\partial_{\zeta^-}\bar A\Big) - \\
&\quad - \bar\theta(\theta\chi)\left(\bar\psi_R\partial_{\xi^-}\bar A + F\psi_L\right) - \bar\chi(\theta\chi)\left(\bar\psi_L\partial_{\zeta^-}\bar A - F\psi_R\right) - \\
&\quad - \bar\theta\bar\chi\theta\Big\{\frac{i}{2}\Big(-C^{11}\partial_{\zeta^-}\psi_L F + C^{10}\partial_{\xi^-}\psi_R F -
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - C^{01}\bar{\psi}_L \partial_{\xi^-}\partial_{\zeta^-}\bar{A} - C^{11}\bar{\psi}_R \partial_{\xi^-}\partial_{\zeta^-}\bar{A}\Big) - \bar{\psi}_L \bar{F}\Big\} - \\
&-\bar{\theta}\bar{\chi}\chi\Big\{\frac{i}{2}\Big(C^{00}\partial_{\xi^-}\psi_R F - C^{10}\partial_{\zeta^-}\psi_L F - \\
&\qquad - C^{01}\bar{\psi}_R \partial_{\xi^-}\partial_{\zeta^-}\bar{A} - C^{00}\bar{\psi}_L \partial_{\xi^-}\partial_{\zeta^-}\bar{A}\Big) + \bar{\psi}_R \bar{F}\Big\} + \\
&+\bar{\theta}\bar{\chi}(\theta\chi)\left(i\bar{\psi}_L \partial_{\zeta^-}\psi_L + i\bar{\psi}_R \partial_{\xi^-}\psi_R - F\bar{F}\right). \qquad \text{(A.9)}
\end{aligned}
$$

From eqn. (A.8), one can notice that all the terms in $R*L$, which depend on $C^{\alpha\beta}$ (Lorentz non invariant) come with opposite signs to the ones in $L*R$ given in eqn. (A.9). However, other terms in $R*L$ independent of $C^{\alpha\beta}$ come with the same sign as in $L*R$. In particular, we notice that the following results hold true when we restrict ourselves to terms which are proportional to $\bar{\theta}\bar{\chi}(\theta\chi)$:

$$
L*R|_{\bar{\theta}\bar{\chi}\theta\chi} = R*L|_{\bar{\theta}\bar{\chi}\theta\chi}\,. \qquad \text{(A.10)}
$$

Using eqns. (A.8) and (A.9), one can also show the following:

$$
\begin{aligned}
L_*^2 * R|_{\bar{\theta}\bar{\chi}\theta\chi} &= L*R*L|_{\bar{\theta}\bar{\chi}\theta\chi} && \text{(A.11)}\\
&= R*L_*^2|_{\bar{\theta}\bar{\chi}\theta\chi}\,, && \text{(A.12)}\\
R_*^2 * L|_{\bar{\theta}\bar{\chi}\theta\chi} &= R*L*R|_{\bar{\theta}\bar{\chi}\theta\chi} && \text{(A.13)}\\
&= L*R_*^2|_{\bar{\theta}\bar{\chi}\theta\chi}\,. && \text{(A.14)}
\end{aligned}
$$

Now, the combination in which $L*R$ appears in the action is $[L*R] = L*R + R*L$, in which the terms proportional to $C^{\alpha\beta}$ cancel out.

Further, using the definition of R given in eqn. (A.3) and the results in eqns. (A.4)–(A.5), the following identities can be proved:

$$
\begin{aligned}
L_*^{2n} * R|_{\bar{\theta}\bar{\chi}\theta\chi} &= L*R*L_*^{2n-1}|_{\bar{\theta}\bar{\chi}\theta\chi} = \cdots = R*L_*^{2n}|_{\bar{\theta}\bar{\chi}\theta\chi}\,, && \text{(A.15)}\\
L_*^{2n+1} * R|_{\bar{\theta}\bar{\chi}\theta\chi} &= L*R*L_*^{2n}|_{\bar{\theta}\bar{\chi}\theta\chi} = \cdots = R*L_*^{2n+1}|_{\bar{\theta}\bar{\chi}\theta\chi}\,, && \text{(A.16)}\\
L_*^{2n} * R_*^2|_{\bar{\theta}\bar{\chi}\theta\chi} &= L_*^{2n-1}*R*L*R|_{\bar{\theta}\bar{\chi}\theta\chi} = \cdots = R_*^2*L_*^{2n}|_{\bar{\theta}\bar{\chi}\theta\chi}\,, && \text{(A.17)}\\
L_*^{2n+1} * R_*^2|_{\bar{\theta}\bar{\chi}\theta\chi} &= R*L_*^{2n}*R*L|_{\bar{\theta}\bar{\chi}\theta\chi} = \cdots = R_*^2*L_*^{2n+1}|_{\bar{\theta}\bar{\chi}\theta\chi}\,. && \text{(A.18)}
\end{aligned}
$$

Below, we give conclusive evidence for these identities.

1. First, we show that for terms with coefficient $\bar{\theta}\bar{\chi}(\theta\chi)$, the following result holds true:

$$
L_*^{2n} * R|_{\bar{\theta}\bar{\chi}\theta\chi} = R*L_*^{2n}|_{\bar{\theta}\bar{\chi}\theta\chi}\,. \qquad \text{(A.19)}
$$

From the definition of R given in eqn. (A.3) and the result for in eqn. (A.4), we can calculate the following:

$$
\begin{aligned}
R*L_*^{2n} = &\left(\frac{1}{4}\det C\right)^{n-1} F^{2n-2}(-2n)\bar{\psi}_L\bar{\psi}_R \times \\
&\times\Big(-i(\theta\chi)\bar{\theta}\psi_L + i(\theta\chi)\bar{\chi}\psi_R + \frac{i}{2}\Big(C^{00}\chi\bar{\theta} + C^{01}\theta\bar{\theta}\Big)\partial_{\xi^-}\bar{A} - \\
&- \frac{i}{2}\Big(C^{11}\theta\bar{\chi} + C^{10}\chi\bar{\chi}\Big)\partial_{\zeta^-}\bar{A} + \bar{\theta}\bar{\chi}\times
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big\{ \frac{1}{2}\Big(C^{11}\theta + C^{10}\chi\Big)\partial_{\zeta^-}\psi_L - \frac{1}{2}\Big(C^{01}\theta + C^{00}\chi\Big)\partial_{\xi^-}\psi_R - i(\theta\chi)\bar{F} - \\
&\quad - \left(\frac{1}{4}\det C\right)\partial_{\xi^-}\partial_{\zeta^-}\bar{A}\Big\}\Big) + \left(\frac{1}{4}\det C\right)^n F^{2n} R\,. \qquad \text{(A.20)}
\end{aligned}
$$

Similarly, one can also obtain:

$$
\begin{aligned}
L_*^{2n} * R = {}& \left(\frac{1}{4}\det C\right)^{n-1} F^{2n-2}(-2n)\bar{\psi}_L\bar{\psi}_R \times \\
&\times \Big(-i(\theta\chi)\bar{\theta}\psi_L + i(\theta\chi)\bar{\chi}\psi_R - \frac{i}{2}\Big(C^{00}\chi\bar{\theta} + C^{01}\theta\bar{\theta}\Big)\partial_{\xi^-}\bar{A} + \\
&\quad + \frac{i}{2}\Big(C^{11}\theta\bar{\chi} + C^{10}\chi\bar{\chi}\Big)\partial_{\zeta^-}\bar{A} + \bar{\theta}\bar{\chi} \times \\
&\quad \times \Big\{ -\frac{1}{2}\Big(C^{11}\theta + C^{10}\chi\Big)\partial_{\zeta^-}\psi_L + \frac{1}{2}\Big(C^{01}\theta + C^{00}\chi\Big)\partial_{\xi^-}\psi_R - i(\theta\chi)\bar{F} - \\
&\quad - \left(\frac{1}{4}\det C\right)\partial_{\xi^-}\partial_{\zeta^-}\bar{A}\Big\}\Big) + \left(\frac{1}{4}\det C\right)^n F^{2n} R\,. \qquad \text{(A.21)}
\end{aligned}
$$

Comparing the results in equations (A.20) and (A.21), one can infer that the identity given eqn. (A.19) holds true. Further, the identity in eqn. (A.19), seems to hold, even for other terms which neither depend on C nor on $\bar{\theta}\bar{\chi}(\theta\chi)$. However, these will not contribute to the action after integration over the grassmannian variables, and hence, are not interesting for our purposes. Now, from the identity (A.19) and using eqns. (A.20) and (A.21) it is possible to see that the following relations also hold true:

$$
\begin{aligned}
L_*^{2n} * R_*^2|_{\bar{\theta}\bar{\chi}\theta\chi} &= R * L_*^{2n} * R|_{\bar{\theta}\bar{\chi}\theta\chi} \\
&= R_*^2 * L_*^{2n}|_{\bar{\theta}\bar{\chi}\theta\chi}\,, \qquad \text{(A.22)} \\
L_*^{2n+1} * R|_{\bar{\theta}\bar{\chi}\theta\chi} &= L * R * L_*^{2n}|_{\bar{\theta}\bar{\chi}\theta\chi}\,. \qquad \text{(A.23)}
\end{aligned}
$$

2. Next, making use of the result given in eqn. (A.5), one can directly compute the following quantities:

$$
\begin{aligned}
R * L_*^{2n+1} &= \left(\frac{1}{4}\det C\right)^n F^{2n-1}\Big[-2ni\bar{\psi}_L\bar{\psi}_R R + F\Big(R * L\Big)\Big] \qquad \text{(A.24)} \\
L_*^{2n+1} * R &= \left(\frac{1}{4}\det C\right)^n F^{2n-1}\Big[-2ni\bar{\psi}_L\bar{\psi}_R R + F\Big(L * R\Big)\Big]\,. \qquad \text{(A.25)}
\end{aligned}
$$

In general, the quantities in eqns. (A.24) and (A.25) are not equivalent. However, for terms with coefficient $\bar{\theta}\bar{\chi}(\theta\chi)$, the result in eqn. (A.10) can be used to see the equivalence of second terms and hence, the following identity can be established:

$$
L_*^{2n+1} * R|_{\bar{\theta}\bar{\chi}\theta\chi} = R * L_*^{2n+1}|_{\bar{\theta}\bar{\chi}\theta\chi}\,. \qquad \text{(A.26)}
$$

3. Similarly, from eqn. (A.5) one can also calculate the following quantities:

$$
\begin{aligned}
R_*^2 * L_*^{2n+1} &= \left(\frac{1}{4}\det C\right)^n F^{2n-1}\Big[-2ni\bar{\psi}_L\bar{\psi}_R R_*^2 + F\Big(R_*^2 * L\Big)\Big] \qquad \text{(A.27)} \\
L_*^{2n+1} * R_*^2 &= \left(\frac{1}{4}\det C\right)^n F^{2n-1}\Big[-2ni\bar{\psi}_L\bar{\psi}_R R_*^2 + F\Big(L * R_*^2\Big)\Big]\,. \qquad \text{(A.28)}
\end{aligned}
$$

Now, using the result given in eqn. (A.14) in eqn. (A.27), the following equivalence can be shown for terms with coefficient $\bar\theta\bar\chi(\theta\chi)$:

$$R_*^2 * L_*^{2n+1}|_{\bar\theta\bar\chi\theta\chi} = L_*^{2n+1} * R_*^2|_{\bar\theta\bar\chi\theta\chi}\,. \tag{A.29}$$

Moreover, from eqns. (A.24) and (A.26), one can also derive the following identities, when restricted to terms with coefficient $\bar\theta\bar\chi(\theta\chi)$:

$$R_*^2 * L_*^{2n+1}|_{\bar\theta\bar\chi\theta\chi} = R * L_*^{2n+1} * R|_{\bar\theta\bar\chi\theta\chi} \tag{A.30}$$

$$L * R * L_*^{2n+1}|_{\bar\theta\bar\chi\theta\chi} = L_*^{2n+2} * R|_{\bar\theta\bar\chi\theta\chi}\,. \tag{A.31}$$

4. Now, we proceed to establish few more identities, which will ultimately lead to the proof of the result given in eqn. (A.1). First, using the result for L_*^{2m} given in eqn. (A.4) and also the result for $R * L_*^{2n}$ given in eqn. (A.20), we can compute the following quantity:

$$\begin{aligned} L_*^{2m} * (R * L_*^{2n})|_{\bar\theta\bar\chi\theta\chi} = {} & \Big(\frac{1}{4}\det C\Big)^{m-1} F^{2m-2} \times \\ & \times \Bigg[(-2m)\Big(\frac{1}{4}\det C\Big)^{n} F^{2n}\bar\psi_L\bar\psi_R(-i\bar F) + \Big(\frac{1}{4}\det C\Big)F^2 \times \\ & \quad \times (-i\bar F)\Big(\frac{1}{4}\det C\Big)^{n-1} F^{2n-2}(-2n\bar\psi_L\bar\psi_R) + \Big(\frac{1}{4}\det C\Big) \times \\ & \quad \times F^2\Big(\frac{1}{4}\det C\Big)^{n} F^{2n}\partial_{\xi^-}\partial_{\zeta^-}\bar A\Bigg]. \end{aligned} \tag{A.32}$$

Independently, from the definition of L_*^{2n} in eqn. (A.4), one can also arrive at the following relation:

$$\begin{aligned} L_*^{2m+2n} * R|_{\bar\theta\bar\chi\theta\chi} = {} & \left(\frac{1}{4}\det C\right)^{m+n-1} F^{2(m+n)-2}(-2m-2n)(-i\bar F\bar\psi_L\bar\psi_R) + \\ & + \left(\frac{1}{4}\det C\right)^{m+n} F^{2(m+n)}\partial_{\xi^-}\partial_{\zeta^-}\bar A\,. \end{aligned} \tag{A.33}$$

Comparing eqns. (A.32) and (A.33), one can see that the following holds true:

$$L_*^{2m} * (R * L_*^{2n})|_{\bar\theta\bar\chi\theta\chi} = L_*^{2m+2n} * R|_{\bar\theta\bar\chi\theta\chi}\,. \tag{A.34}$$

5. In a similar way, using the result derived in eqn. (A.5) for L_*^{2n+1}, one can explicitly deduce that:

$$\begin{aligned} L_*^{2m+2n+1} * R|_{\bar\theta\bar\chi\theta\chi} = {} & \Bigg[\left(\frac{1}{4}\det C\right)^{m+n} F^{2(m+n)-1}\left(-2(m+n)i\bar\psi_L\bar\psi_R\partial_{\xi^-}\partial_{\zeta^-}\bar A\right) + \\ & + F\left(i\bar\psi_L\partial_{\zeta^-}\psi_L + i\bar\psi_R\partial_{\xi^-}\psi_R - F\bar F\right)\Bigg]. \end{aligned} \tag{A.35}$$

On the other hand, using eqn. (A.4), one can also show that:

$$\begin{aligned}
&L_*^{2m} * (R * L_*^{2n+1})|_{\bar\theta\bar\chi\theta\chi} = \\
&= \left(\frac{1}{4}\det C\right)^n F^{2n-1}\left[-2ni\bar\psi_L\bar\psi_R L_*^{2m} * R + F.L_*^{2m} * R * L\right]|_{\bar\theta\bar\chi\theta\chi} \\
&= \left(\frac{1}{4}\det C\right)^n F^{2n-1} \times \\
&\quad \times \Bigg[(-2in)\bar\psi_L\bar\psi_R\left(\frac{1}{4}\det C\right)^m F^{2m}\partial_{\xi^-}\partial_{\zeta^-}\bar A + \\
&\qquad + F\left(\frac{1}{4}\det C\right)^{m-1} F^{2m-1} \times \\
&\qquad \times \left\{(-2m)\bar\psi_L\bar\psi_R i\left(\frac{1}{4}\det C\right) F\partial_{\xi^-}\partial_{\zeta^-}\bar A + \right. \\
&\qquad\quad \left. + \left(\frac{1}{4}\det C\right) F^2\left(i\bar\psi_L\partial_{\zeta^-}\psi_L + i\bar\psi_R\partial_{\xi^-}\psi_R - F\bar F\right)\right\}\Bigg]. \qquad (A.36)
\end{aligned}$$

Thus, comparing eqns. (A.35) and (A.36)we have:

$$L_*^{2m} * (R * L_*^{2n+1})|_{\bar\theta\bar\chi\theta\chi} = L_*^{2m+2n+1} * R|_{\bar\theta\bar\chi\theta\chi}\,. \qquad (A.37)$$

6. Now, using the definition of L_*^{2n} given in eqn. (A.4), one can deduce that:

$$\begin{aligned}
L_*^{2(m+n+1)} * R|_{\bar\theta\bar\chi\theta\chi} &= \left(\frac{1}{4}\det C\right)^{m+n} F^{2(m+n)} \times \qquad (A.38)\\
&\quad \times \left[-2(m+n+1)i\bar\psi_L\bar\psi_R(-i\bar F) + \left(\frac{1}{4}\det C\right) F^2\partial_{\xi^-}\partial_{\zeta^-}\bar A\right].
\end{aligned}$$

On the other hand, using the definition of L_*^{2n+1} given in eqn. (A.5), one has:

$$\begin{aligned}
L_*^{2m+1} * R * L_*^{2n+1}|_{\bar\theta\bar\chi\theta\chi} &= \left(\frac{1}{4}\det C\right)^{m+n} F^{2(m+n)-2} \times \\
&\quad \times \Big[-2mi\bar\psi_L\bar\psi_R F R * L - 2ni\bar\psi_L\bar\psi_R F L * R + \\
&\qquad + F^2 L * R * L\Big]|_{\bar\theta\bar\chi\theta\chi} \\
&= \left(\frac{1}{4}\det C\right)^{m+n} F^{2(m+n)-2} \times \\
&\quad \times \Bigg[-2mi\bar\psi_L\bar\psi_R F(-F\bar F) - 2ni\bar\psi_L\bar\psi_R F(-F\bar F) + F^2 \times \\
&\qquad \times \left\{\left(\frac{1}{4}\det C\right) F^2\partial_{\xi^-}\partial_{\zeta^-}\bar A + 2i\bar\psi_L\bar\psi_R\bar F\right\}\Bigg]. \qquad (A.39)
\end{aligned}$$

Comparing eqns. (A.38) and (A.39), one can see that:

$$L_*^{2n+1} * R * L_*^{2n+1}|_{\bar\theta\bar\chi\theta\chi} = L_*^{2(m+n+1)} * R|_{\bar\theta\bar\chi\theta\chi}\,. \qquad (A.40)$$

7. To show that:

$$L_*^{2m} * R * L_*^{2n} * R * L_*^{2p}|_{\bar\theta\bar\chi\theta\chi} = L_*^{2(m+n+p)} * R_*^2|_{\bar\theta\bar\chi\theta\chi}\,, \tag{A.41}$$

we directly calculate the following from the definitions of L_*^{2p} and R given in eqns. (A.4) and (A.3):

$$\begin{aligned}
&L_*^{2m} * R * L_*^{2n} * R * L_*^{2p}|_{\bar\theta\bar\chi\theta\chi} = \\
&= \left(\frac{1}{4}\det C\right)^{m+n+p-3} F^{2(m+n+p)-6}\left[-2m(\theta\chi)\bar\psi_L\bar\psi_R + \left(\frac{1}{4}\det C\right)F^2\right] * R \times \\
&\times * \left[-2n(\theta\chi)\bar\psi_L\bar\psi_R + \left(\frac{1}{4}\det C\right)F^2\right] * R * \left[-2p(\theta\chi)\bar\psi_L\bar\psi_R + \left(\frac{1}{4}\det C\right)F^2\right] \\
&= \left(\frac{1}{4}\det C\right)^{m+n+p-3} F^{2(m+n+p)-6}\Bigg[-2(m+n+p)\left(\frac{1}{4}\det CF^2\right)^2 \bar\psi_L\bar\psi_R \times \\
&\qquad \times(-2\psi_L\psi_R) + 2\left(\frac{1}{4}\det CF^2\right)^3 \partial_{\xi^-}\partial_{\zeta^-}\bar A\Bigg].
\end{aligned} \tag{A.42}$$

On the other hand, one can also calculate the following directly from eqn. (A.5):

$$\begin{aligned}
L_*^{2(m+n+p)} * R_*^2|_{\bar\theta\bar\chi\theta\chi} &= \left(\frac{1}{4}\det C\right)^{m+n+p-1} F^{2(m+n+p)-2} \times \\
&\times\left[-2(m+n+p)\bar\psi_L\bar\psi_R(-2\psi_L\psi_R) + 2\left(\frac{1}{4}\det CF^2\right)\partial_{\xi^-}\partial_{\zeta^-}\bar A\right].
\end{aligned} \tag{A.43}$$

8. The following identity can be shown in an identical fashion:

$$L_*^{2m+1} * R * L_*^{2n} * R * L_*^{2p}|_{\bar\theta\bar\chi\theta\chi} = L_*^{2(m+n+p)+1} * R_*^2|_{\bar\theta\bar\chi\theta\chi}\,. \tag{A.44}$$

9. Next, using the definitions of L_*^{2p}, L_*^{2n+1} and R given in eqn. (A.5), (A.4) and (A.3) one can directly calculate the following:

$$\begin{aligned}
&L_*^{2m+1} * R * L_*^{2n+1} * R * L_*^{2p}|_{\bar\theta\bar\chi\theta\chi} = \\
&= \left(\frac{1}{4}\det C\right)^m F^{2m-1}\left[-2im\bar\psi_L\bar\psi_R + F.L\right] * R * \\
&\quad * \left(\frac{1}{4}\det C\right)^n F^{2n-1}\left[-2in\bar\psi_L\bar\psi_R + F.L\right] * R \times \\
&\quad \times * \left(\frac{1}{4}\det C\right)^{p-1} F^{2p-2}\left[-2p(\theta\chi)\bar\psi_L\bar\psi_R + \left(\frac{1}{4}\det C\right)F^2\right] \\
&= \left(\frac{1}{4}\det C\right)^{m+n+p-1} F^{2(m+n+p)-4} \times \\
&\quad \times\Bigg[-2im\bar\psi_L\bar\psi_R\left(\frac{1}{4}\det C\right)F^3 R * L * R - -2in\bar\psi_L\bar\psi_R\left(\frac{1}{4}\det C\right)F^3 L * R * R + \\
&\qquad + F^2 L * R * L * R(-2p(\theta\chi)) + \left(\frac{1}{4}\det C\right)F^2 L * R * L * R\Bigg]\Bigg|_{\bar\theta\bar\chi\theta\chi}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{4}\det C\right)^{m+n+p-1} F^{2(m+n+p)-4} \times \\
&\times \Bigg[-2im\bar{\psi}_L\bar{\psi}_R \left(\frac{1}{4}\det C\right) F^3 (2i)\psi_L\psi_R F - 2in\bar{\psi}_L\bar{\psi}_R(2i)\psi_L\psi_R F \left(\frac{1}{4}\det C\right) F^3 + \\
&\quad + F^2 \left(-2.\left(\frac{1}{4}\det C\right) F^2\psi_L\psi_R\right)(-2p) + \left(\frac{1}{4}\det C\right) F^2 2 \times \\
&\quad \times \left\{\left(\frac{1}{4}\det C\right) F^2 \partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A} + 2\bar{\psi}_L\bar{\psi}_R\psi_L\psi_R\right\}\Bigg]. \qquad \text{(A.45)}
\end{aligned}
$$

In deriving the above identity, we have made use of the following results:

$$
\begin{aligned}
L * R * L * R|_{\bar{\theta}\bar{\chi}\theta\chi} &= R_*^2 * L_*^2|_{\bar{\theta}\bar{\chi}\theta\chi} \\
&= 2\left\{\left(\frac{1}{4}\det C\right) F^2 \partial_{\xi^-}\bar{A}\partial_{\zeta^-}\bar{A} + 2\bar{\psi}_L\bar{\psi}_R\psi_L\psi_R\right\}, \\
L * R * L * R|_{\bar{\theta}\bar{\chi}} &= -2.\left(\frac{1}{4}\det C\right) F^2\psi_L\psi_R\,. \qquad \text{(A.46)}
\end{aligned}
$$

Thus, one ends up with:

$$
\begin{aligned}
L_*^{2m+1} * R * L_*^{2n+1} * R * L_*^{2p}|_{\bar{\theta}\bar{\chi}\theta\chi} &= \left(\frac{1}{4}\det C\right)^{m+n+p} F^{2(m+n+p)} \times \\
&\quad \times \Bigg[4(m+n+p+1)\bar{\psi}_L\bar{\psi}_R\psi_L\psi_R + \\
&\quad + \left(\frac{1}{4}\det C\right) F^2 2\partial_{\xi^-}\partial_{\zeta^-}\bar{A}\Bigg]|_{\bar{\theta}\bar{\chi}\theta\chi} \\
&= L_*^{2(m+n+p)+2} * R^2|_{\bar{\theta}\bar{\chi}\theta\chi}. \qquad \text{(A.47)}
\end{aligned}
$$

10. One can now go on and rigorously show that the following identity holds true as well (when restricted to terms with coefficient $\bar{\theta}\bar{\chi}(\theta\chi)$):

$$
L_*^{2m+1} * R * L_*^{2n+1} * R * L_*^{2p+1}|_{\bar{\theta}\bar{\chi}\theta\chi} = L_*^{2(m+n+p)+3} * R_*^2|_{\bar{\theta}\bar{\chi}\theta\chi}\,, \qquad \text{(A.48)}
$$

In this manner, one finally arrives at the most general result that takes the form:

$$
L_*^m * R * L_*^n * R * L_*^p|_{\bar{\theta}\bar{\chi}\theta\chi} = L_*^{(m+n+p)} * R_*^2|_{\bar{\theta}\bar{\chi}\theta\chi},\,, \qquad \text{(A.49)}
$$

where m, n and p are all integers.

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